Birational geometry	Fall 2023
Lecture 4 — 10,14, 2023	
	Scribe: Yi Li

1 Overview

In the last lecture we prove the adjunction formula and varies generalization(including adjunction formula for normal variety, subadjunction formula for DLT pair, and Kawamata's subadjunction formula).

In this lecture we begin to discuss the intersection number and numerical geometry. The main theorem that we will prove in this lecture is the Nakai-Moishezon-Kleiman's ampleness criterion.

2 The intersection number

We begin by defining the intersection number. Let X be an algebraic variety, $D_1, \cdots, D_r$ Cartier divisors on $X, \mathscr{F}$ a coherent $\mathscr{O}_X$-Module such that Supp $\mathscr{F}$ is complete of r-dimension. Then there exists a polynomial $P(z_1, \cdots, z_r)$ of degree $\leq n$ with coefficients in $\mathbb{Q}$ such that

$$\begin{aligned} & P(m_1, \cdots, m_r) \\ & = \chi\left(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}\left(m_1 D_1 + \cdots + m_r D_r\right)\right) \\ & := \sum_i (-1)^i \dim H^i\left(X, \mathscr{F} \otimes \mathscr{O}\left(m_1 D_1, + \cdots + m_r D_r\right)\right), \end{aligned}$$

for every $m_1, \cdots, m_r \in \mathbb{Z}$

Definition 1. *When $r \geq n$, the coefficient of the monomial $z_1 \cdots z_r$ in $P(z_1, \cdots, z_r)$ is called the intersection number of $D_1, \cdots, D_r$ with respect to $\mathscr{F}$ and denoted by $(D_1, \cdots, D_r; \mathscr{F})$.*

In particular, if $\mathscr{F} = \mathscr{O}_Y$ for a complete closed subvariety $Y \subset X$, we denote $(D_1, \cdots, D_r; \mathscr{F})$ by $(D_1, \cdots, D_r; Y)$. If, moreover, $Y = X$, we denote it simply by $(D_1, \cdots, D_r)$. If D_i all coincide with D, then we denote it by D^r.

The second definition is more comprehensive.

Definition 2. *Let $D_1, \ldots, D_k$ be Cartier divisor, and V be subvarierty of dimension k then we can define the intersection numer*

$$(D_1, \ldots, D_k, V) := (c_1(D_1) \cup c_1(D_2) \ldots \cup c_1(D_k)) \cdot [V],$$

where $[V] \in H_{2k}(X)$ is the fundamental calss associated to V, and $\cdot$ is the perfect bilinear pairing

$$H^{2k}(X, \mathbb{Z}) \times H_{2k}(X, \mathbb{Z}) \to \mathbb{Z}$$

we can use wedge product in deRham cohomology in replace of the cup product in singular cohomology when X is a complex manifold.

3 Neron-Severi group and Picard group

In this section I will define the Neron-Severi group $N^1(X)$ and Picard group.

I will prove in this section that Neron-Severi group is a finite rank free Abelian group and show the duality between the Neron-Severi group and the group of numerical equivalence class of curves $N_1(X)$.

Definition 3. *Let X be a projective(or complete) scheme over $\mathbb{C}$, The Néron-Severi group of X is the group*

$$N^1(X) = \operatorname{Div}(X)/\operatorname{Num}(X),$$

of numerical equivalence classes of (Cartier) divisors on X. Where $\operatorname{Num}(X)$ is the subgroup of the Cartier divisor group that $D \equiv 0$.

Néron-Severi group is indeed finite rank and torsion free:

Theorem 4. *The Neron-Severi group $N^1(X)$ is a free abelian group of finite rank.*

Proof. A divisor D on X determines a cohomology class

$$[D]_{\text{hom}} = c_1(\mathcal{O}_X(D)) \in H^2(X;\mathbf{Z}),$$

and if $[D]_{\text{hom}} = 0$ then evidently D is numerically trivial. Therefore the group $\operatorname{Hom}(X)$ of cohomologically trivial Cartier divisors is a subgroup of $\operatorname{Num}(X)$. It follows that $N^1(X)$ is a quotient of a subgroup of $H^2(X;\mathbf{Z})$, and in particular is finitely generated. For torsion freness one needs to use Corollary 1.4.38 in Lazarsfeld's book. □

If we define $Z_1(X) = \{\sum_i^n a_iC_i \mid a_i \in \mathbb{Z}\}$ and two elements $C_1 \equiv C_2 \in Z_1(X)$ iff $C_1 \cdot D = C_2 \cdot D$ for all prime divisor D, and $N_1(X) = Z_1(X)/\equiv$. Then we can induce from the intersection pairing $\operatorname{Div} \otimes Z_1(X) \to \mathbb{Z}$ a perfect pairing

$$N_1(X) \times N^1(X) \to \mathbb{Z}$$

.

Since $N^1(X)$ is finite rank free abelian group, one has

$$\dim N^1(X)_{\mathbb{R}} = \dim N^1(X)_{\mathbb{R}} < \infty$$

Therefore using intersection number we reduce the infinite dimension space $\operatorname{Div}(X)_{\mathbb{R}}$ into a finite dimension vector space $N^1(X)$. Mori realized that one can use the cone geometry in this finite dimension vector space $N^1(X)_{\mathbb{R}}$ to get some information about a given variety. To do this we need the concept of positivities.

4 Nakai-Moishezon-Kleiman's ampleness criterion

Here is the main theorem that will be proved in today's lecture is

Theorem 5. *Let $L = \mathcal{O}(D)$ be a line bundle on a projective variety X. Then L is ample if and only if*

$$\int_V c_1(L)^{\dim(V)} > 0$$

for every positive-dimensional irreducible subvariety $V \subseteq X$.

Before proving this result, let me make few remarks: Campana and Peternell [1] extend to result to $\mathbb{R}$-Cartier divisor on the projective scheme, In a groundbreaking paper, Demailly and Paun [2], proved a vast generalization of this result, which holds for all real $(1,1)$ classes on a compact Kähler manifold. Recently Fujino and Miyamoto [3] generalize the result of Campana and Peternell to the complete schemes.

Proof. Suppose that D is ample.

Then mD is very ample for some $m > 0$. Let $\phi : X \longrightarrow \mathbb{P}^N$ be the corresponding embedding. Then $mD = \phi^* H$, where H is a hyperplane in $\mathbb{P}^N$. Then

$$(mD)^k \cdot V = H^k \cdot \phi(V) > 0,$$

where the first equality due to projection formula, since intersecting $\phi(V)$ with H^k corresponds to intersecting V with a linear space of dimension $N - k$. And this is the degree of $\phi(V)$ in projective space.

Now we prove the converse direction: assuming the positivity of the intersection numbers appearing in the Theorem, we prove that $\mathcal{O}(D)$ is ample. The result being clear if $\dim X = 1$, indeed it follows from the Riemann-Roch theorem(For the details, we refer the reader to Qing Liu's book Proposition 5.5).

we put $n = \dim X$ and assume inductively that the Theorem is known for all schemes of dimension $\leq n-1$. It is convenient at this point to switch to additive notation, so write $L = \mathcal{O}_X(D)$ for some divisor D on X.

We assert first that

$$H^0\left(X, \mathcal{O}_X(mD)\right) \neq 0 \text{ for } m \gg 0.$$

In fact, asymptotic Riemann-Roch gives to begin with that

$$\chi\left(X, \mathcal{O}_X(mD)\right) = m^n \frac{(D^n)}{n!} + O\left(m^{n-1}\right),$$

and $(D^n) = \int_X c_1(L)^n > 0$ by assumption. Now write $D \sim A - B$ as a difference of very ample effective divisors A and B (Every Cartier divisor on projective scheme can be written as difference of two very ample divisors which are linear equivalent to some effective divisor). We have two exact sequences:

$$0 \longrightarrow \mathcal{O}_X(mD - B) \xrightarrow{\cdot A} \mathcal{O}_X((m+1)D) \longrightarrow \mathcal{O}_A((m+1)D) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_X(mD - B) \xrightarrow{\cdot B} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_B(mD) \longrightarrow 0$$

By induction, $\mathcal{O}_A(D)$ and $\mathcal{O}_B(D)$ are ample. Consequently the higher cohomology of each of the two sheaves on the right vanishes when $m \gg 0$. So we find that if $m \gg 0$, then

$$H^i\left(X, \mathcal{O}_X(mD)\right) = H^i\left(X, \mathcal{O}_X(mD - B)\right) = H^i\left(X, \mathcal{O}_X((m+1)D)\right)$$

for $i \geq 2$. In other words, if $i \geq 2$ then the dimensions $h^i(X, \mathcal{O}_X(mD))$ are eventually constant. Therefore

$$\chi(X, \mathcal{O}_X(mD)) = h^0(X, \mathcal{O}_X(mD)) - h^1(X, \mathcal{O}_X(mD)) + C$$

for some constant C and $m \gg 0$. So it follows that $H^0(X, \mathcal{O}_X(mD))$ is non-vanishing when m is sufficiently large, as asserted. Since D is ample if and only if mD is, there is no loss in generality in replacing D by mD. Therefore we henceforth suppose that D is effective.

We next prove that $\mathcal{O}_X(mD)$ is generated by its global sections if $m \gg 0$. As D is assumed to be effective this is evidently true away from $\mathrm{Supp}(D)$, indeed one has $\mathcal{O}_{X,x}(mD) \cong \mathcal{O}_{X,x}$, since for effective divisor $D \geq 0$ there is a inclusion of sheaf $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(mD)$for $n \geq 1$, therefore $H^0(X, \mathcal{O}(mD)) \supset H^0(X, \mathcal{O}_X)$. Which shows the globally generated of $\mathcal{O}_X(mD)$ at $x \notin D$.

so the issue is to show that no point of D is a base point of the linear series $|\mathcal{O}_X(mD)|$. Consider to this end the exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \xrightarrow{\cdot D} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0$$

As before, $\mathcal{O}_D(D)$ is ample by induction. Consequently $\mathcal{O}_D(mD)$ is globally generated and $H^1(X, \mathcal{O}_D(mD)) = 0$ for $m \gg 0$. It then follows that the natural homomorphism

$$H^1(X, \mathcal{O}_X((m-1)D)) \longrightarrow H^1(X, \mathcal{O}_X(mD))$$

is surjective for every $m \gg 0$. The spaces in question being finite dimensional, the maps must actually be isomorphisms for sufficiently large m. Therefore the restriction mappings

$$H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(X, \mathcal{O}_D(mD))$$

are surjective for $m \gg 0$(that is we can lift the section). But since $\mathcal{O}_D(mD)$ is globally generated, it follows that no point of $\mathrm{Supp}(D)$ is a base-point of $|mD|$, as required.

Finally, the amplitude of $\mathcal{O}_X(mD)$. Since $\mathcal{O}_X(mD)$ is base point free, there is a morphism

$$\phi_{|mD|} : X \to \mathbb{P}^N$$

such that $mD = \phi^* H$ for some hyperplane divisor. I claim this morphism is finite, for if it contracts some curves then by the projection formula one has $\phi^* H \cdot C = 0$, on the other hand one has by the assumption one has $mD \cdot C > 0$ contradiction. Therefore mD must be an ample divisor. □

References

[1] F. Campana, T. Peternell, *Algebraicity of the ample cone of projective varieties*, J. Reine Angew. Math. 407 (1990) 160–166.

[2] Demailly, Jean-Pierre and Paun, Mihai, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Annals of mathematics. (2004) 1247–1274.

[3] Fujino, Osamu and Miyamoto, Keisuke *Nakai–Moishezon ampleness criterion for real line bundles*, Mathematische Annalen. 385 (2023) 459–470.