Birational geometry reading seminars Fall 2023

Lecture 5 — 11,2, 2023

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1 Overview

Today we continue the discussion of Birkar's lecture notes on birational geometry. Last time we discuss the intersection number, NS group, and Nakai-Moishezon criteron. One important theorem that haven't been proved last time is the Kleiman's ampleness criterion. After that we will study the relation between semiample divisor and contraction morphism, we will show that semiample divisor will induce a contraction morphism, and conversely a contraction morphism will pull back an ample divisor to a semiample divisor. Finally, we will provides some examples about the Mori cone.

2 About $\mathbb{Q}, \mathbb{R}$-coefficient ampleness and nefness

2.1 Ample $\mathbb{Q}$ divisors

Recall the definition of $\mathbb{Q}$ and $\mathbb{R}$-coefficient nefness and ampleness

Definition 1. *A $\mathbb{Q}$-Cartier-divisor$D \in \operatorname{Div}_{\mathbf{Q}}(X)$ is ample if any one of the following three equivalent conditions is satisfied:*

(i)D is of the form $D = \sum c_i A_i$ where $c_i > 0$ is a positive rational number and A_i is an ample Cartier divisor.

(ii) There is a positive integer $r > 0$ such that $r \cdot D$ is integral and ample.

(iii) D satisfies the statement of Nakai's criterion, i.e.

$$\left(D^{\dim V} \cdot V\right) > 0$$

for every irreducible subvariety $V \subseteq X$ of positive dimension.

Let's prove the equivalence

Proof. (i) implies (ii) since $c_i = m_i/n_i$ then taking out the common factor n_i the sum become

$$\frac{1}{n_1 n_2 \ldots n_k} \sum m_i A_i$$

with $\sum m_i A_i$ being ample divisor.

(ii) implies (i) if $r \cdot D$ is integral ample then $1/r(r \cdot D)$ is the form that we want.

(ii) implies (iii) since $r \cdot D$ is integral ample by the Nakai-criterion for the integral divisor we have

$$(r \cdot D)^{\dim V} \cdot V > 0 \implies D^{\dim V} \cdot V > 0$$

(iii) implies (ii) again if $D^{\dim V} \cdot V > 0$ since D is a $\mathbb{Q}$-divisor rD is a integral divisor, therefore by linearity:

$$(rD)^{\dim V} \cdot V = r^{\dim} D^{\dim V} \cdot V > 0$$

□

2.2 Ample $\mathbb{R}$-divisors

For the $\mathbb{R}$-coefficient Cartier divisor however we only have the following definition

Definition 2. *Assume that X is complete. An $\mathbb{R}$-divisor D on X is ample if it can be expressed as a finite sum*

$$D = \sum c_i A_i$$

where $c_i > 0$ is a positive real number and A_i is an ample Cartier divisor.

2.3 Nef $\mathbb{Q}$(or $\mathbb{R}$) divisors

For the $\mathbb{Q}$ and $\mathbb{R}$ nef divisor, the standard definition are slightly different.

Definition 3. *Let X be a complete variety or scheme. A line bundle L on X is numerically effective or nef if*

$$\int_C c_1(L) \geq 0$$

for every irreducible curve $C \subseteq X$. Similarly, a Cartier divisor D on X (with $\mathbf{Z}, \mathbf{Q}$ or $\mathbf{R}$ coefficients) is nef if

$$(D \cdot C) \geq 0$$

for all irreducible curves $C \subset X$.

It's not clear whether the standard definition for nef $\mathbb{R}$-divisor is equivalent to being positive linear combination of integral nef divisors.

3 Kleimann's ampleness criterion

Let's first introduct the Kleiman's ampleness criterion

Theorem 4. *Let X be a normal projective, let D be a $\mathbb{Q}$-Cartier divisor(or $\mathbb{R}$-Cartier divisor) on X then*

$$D \textit{ is ample} \iff D \cdot \alpha > 0 \textit{ for any } \alpha \in \overline{NE(X)} - \{0\}$$

Equivalently, choose any norm— on $N_1(X)_{\mathbb{R}}$, and denote by

$$S = \{\gamma \in N_1(X)_{\mathbb{R}} \mid \|\gamma\| = 1\}$$

the "unit sphere" of classes in $\mathrm{N}_1(X)_{\mathbf{R}}$ of length 1 . Then D is ample if and only if

$$(\overline{\mathrm{NE}}(X) \cap S) \subseteq (D_{>0} \cap S)$$

The equivalence is a direct consequence of definition, I will omit it. Also, the following proof is for the $\mathbb{R}$-Cartier divisors, for the $\mathbb{Q}$-Cartier divisors the proof is essentially the same.

Proof. We assume that the condition holds, and show that D is ample.

To this end, consider the linear functional $\phi_D : N_1(X)_{\mathbf{R}} \longrightarrow \mathbf{R}$ determined by intersection with D. Then $\phi_D(\gamma) > 0$ for all $\gamma \in (\overline{\mathrm{NE}}(X) \cap S)$. But $\overline{\mathrm{NE}}(X) \cap S$ is compact, and therefore ϕ_D is bounded away from zero on this set. In other words, there exists a positive real number $\varepsilon > 0$ such that

$$\phi_D(\gamma) \geq \varepsilon \text{ for all } \gamma \in \overline{\mathrm{NE}}(X) \cap S$$

Thus

$$(D \cdot C) \geq \varepsilon \cdot \|C\|$$

for every irreducible curve $C \subseteq X$. On the other hand, choose ample divisors $H_1, \ldots, H_r$ on X whose classes form a basis of $N^1(X)_{\mathbf{R}}$. Then $\|\cdot\|$ is equivalent to the "taxicab" norm(the reason we require H_i to be the basis of $N^1(X)_{\mathbb{R}}$ is that it makes the definition satisfies the axim of a norm on a vector space)

$$\|\gamma\|_{\text{taxi}} = \sum |(H_i \cdot \gamma)| .$$

Setting $H = \sum H_i$ it therefore follows from (*) that for suitable $\varepsilon' > 0$:

$$(D \cdot C) \geq \varepsilon' \cdot (H \cdot C)$$

for every irreducible curve $C \subseteq X$, which will clearly implies the ampleness of D.

Conversely, if D is ample, we have

$$D \cdot \gamma > 0, \quad \forall \gamma \in \mathrm{NE}(X) \setminus \{0\}$$

(in this step the ample implies the intersection number being positive is clear for both $\mathbb{Q}$ and $\mathbb{R}$ divisors, what is non trivial is the converse implication in the Nakai-Moishezon)

therefore taking closure $D \cdot \gamma \geq 0, \quad \forall \gamma \in \overline{\mathrm{NE}(X)} \setminus \{0\}$ if $\gamma \cdot D = 0$ since γ is not numerically zero element in the Mori cone, there exist some divisor $M \cdot \gamma < 0$ then for sufficient large N we have $ND + M$ is ample therefore

$$0 \leq (M + ND) \cdot \gamma = M \cdot \gamma < 0$$

a contradiction. This finish the proof of the theorem.

□

Since needed in what follows, we prove the relative version Kleiman's ampleness criterion

Theorem 5. *Let $\pi : X \to Y$ be a projective morphism between complex algebraic varieties, Then we have*

$$\mathrm{Amp}(X/Y) = \left\{\zeta \in N^1(X/Y)_{\mathbb{R}} \mid \zeta > 0 \text{ on } \overline{\mathrm{NE}}(X/Y)\backslash\{0\}\right\}$$

Proof. Since being ample is a open condition, to see the ample cone contains in the RHS is the same as the proof for the absolute version.

Conversely, if $\zeta \in$ RHS we want to show it's ample on each fiber X_s (and therefore it's relative ample).

Indeed $\Gamma \in \overline{\mathrm{NE}(X_s)} \subset N^1(X_s)$ is not numerical trival then $\Gamma \in \overline{\mathrm{NE}(X/Y)}$ is not numerical trivial also. To prove this, we use the absolute version Kleiman's ampleness criterion. Choose an relative ample divisor A, it's ample on X_s by definition, therefore $A \cdot_{X_s} \Gamma = A \cdot_X \Gamma > 0$.

Then by our assumption $\zeta \cdot \Gamma > 0$ for any $\Gamma \in \overline{\mathrm{NE}}(X/Y)\backslash\{0\}$ will implies $\zeta \cdot \Gamma > 0$ for any $\Gamma \in \overline{\mathrm{NE}}(X_s)\backslash\{0\}$. □

4 Contraction morphism

We will introduce the semiample fibration theorem in this section

Theorem 6. *Let X be a normal projective variety, and let L be a semiample bundle on X. Then (1) There is an algebraic fibre space(contraction)*

$$\phi : X \longrightarrow Y$$

(2) Having the property that for any sufficiently large integer $k \in M(X, L)$:

$$Y_k = Y \text{ and } \phi_k = \phi.$$

where (3) Furthermolre there is an ample line bundle A on Y such that $f^*A = L^{\otimes f}$, where $f = f(L)$ is the exponent of $M(X, L)$. More roughly speaking, one has

$$L \sim_{\mathbb{Q}} f^*A'$$

for some ample $\mathbb{Q}$-Cartier divisor A'.

For the proof, we refer the reader to Lazarsfeld Theorem 2.1.26.

Conversely given a contraction morphism $f : Y \to X$, we can define a semiample divisor by pull back some ample divisor on X, say $A = f^*H$. Since pull back of semiample divisor is semiample, A is semiample.

For this contraction morphism, a curve $C \subset Y$ is contracted by f iff the intersection $A \cdot C = 0$, we denote $A^{\perp} = \{\gamma \in N^1(X)_{\mathbb{R}} \mid \gamma \cdot A = 0\}$ and $F_A = A^{\perp} \cap N^1(X)_{\mathbb{R}}$

Proof. If C is contracted to a point, by definition $f_*(C) = 0$ by the projection formula $f^*(C) \cdot H = A \cdot C = 0$.

Conversely if $A \cdot C = 0$ it implies $H \cdot f_*C = 0$ if C is not contracted by f, f_*C should has positive intersection with H. □

We claim the following result

Claim 7. *A is numerical trivial on some extreme face $NE(f)$, which is the subcone that generated by the curves being contracted by f.*

Proof. And by the observation above, we have A is numerical trivial on $\mathrm{NE}(f)$.

Only needs to show that $\mathrm{NE}(f)$ is extreme face. Let $a = \sum a_i [C_i]$ and $a' = \sum a'_j \left[C'_j\right]$ be elements of $\mathrm{NE}(X)$, where a_i and a'_j are positive real numbers. If $a+a'$ is in $\mathrm{NE}(\pi)$, there exists a decomposition

$$\sum a_i [C_i] + \sum a'_j [C'_j] = \sum a''_k [C''_k]$$

where the C''_k are irreducible curves contracted by π and the a''_k are positive. Applying π_*, we get $\sum a_i \pi_* [C_i] + \sum a'_j \pi_* \left[C'_j\right] = 0$ in $N_1(Y)_{\mathbf{R}}$. Since Y is projective, the C_i and C'_j must be contracted by π hence a and a' are in $\mathrm{NE}(\pi)$. This proves the claim. □

If we take the closure it becomes the extreme face of Mori cone

Theorem 8. *If taking closure we have*

$$\overline{NE}(f) = \overline{NE(X)} \cap A^{\perp} = F_A$$

therefore the closure of cone of curves being contracted will be a extreme face of the Mori cone.

Proof. By previous discussion we have

$$\overline{\mathrm{NE}}(f) \subset F_A$$

if the inclusion is strict, by basic cone geometry there exist a linear functional seperate then:

There exist $\ell \in (N_1(X)_{\mathbb{R}})^{\vee}$ such that it's strict positive on $\overline{\mathrm{NE}(\pi)} \setminus \{0\}$ but has some $z \in F_A$ such that $\ell(z) < 0$.

Recall that we have $N_1(X)_{\mathbb{R}}^{\vee} \cong N^1(X)_{\mathbb{R}}$ there exist a $\mathbb{R}$ divisor D realize ℓ, by peturbing a little bit, we can assume D is rational coefficient, therefore it can be scaled to integral coefficient divisor.

Therefore by the Kleiman's ampleness criterion we have D is relative ample. Then by the lemma below, for sufficient large $m \gg 0$ we have $mA + D$ is absolute ample, therefore again by Kleiman's criterion, we have

$$0 < (mA + D) \cdot z = D \cdot z < 0$$

contradiction. □

Let's prove the lemma used in the theorem above

Lemma 9. *Let $f : Y \to X$ being contraction morphism, assume D is a relative ample divisor on Y, and H is an ample divisor on X then for sufficient large $m \gg 0$ we have $mf^*H + D$ being ample divisor.*

For the proof we refer the reader to https://stacks.math.columbia.edu/tag/01VG

Theorem 10. *Let X be a normal projective surface and C an irreducible curve which is $\mathbb{Q}$-Cartier as a divisor. If $C^2 < 0$, then C generates an extremal ray of $\overline{NE}(X)$. But if $C^2 > 0$, then the class of C cannot belong to any extremal ray unless $\rho(X) = 1$.*

Proof. Proof. First assume that $C^2 < 0$. Let $\mathcal{C}$ be the subcone of $\overline{NE}(X)$ consisting of those classes α for which $C \cdot \alpha \geq 0$ and let H be the hyperplane in $N_1(X)$ where C is numerically zero. In particular, if C' is any curve other than C, then the class of C' is in $\mathcal{C}$. Moreover, $\overline{NE}(X)$ is nothing but the convex hull of $[C]$ and $\mathcal{C}$. Therefore, $[C]$ generates an extremal ray of $\overline{NE}(X)$ as $[C]$ is on one side of H and $\mathcal{C}$ on the opposite side.

Now the second statement: assume that IC is Cartier and let $f : Y \to X$ be a resolution of singularities. Then, $\left(f^*C^2\right) > 0$ and the RiemannRoch theorem shows that $h^0\left(mIf^*C\right)$ grows like m^2 hence the same holds for $h^0(mIC)$. Pick a general very ample divisor A and consider the exact sequence

$$0 \to H^0(X, mIC - A) \to H^0(X, mIC) \to H^0\left(A, mIC|_A\right)$$

Since A is a smooth curve, $h^0\left(A, mIC|_A\right)$ grows at most like m which shows that $h^0(X, mIC - A)$ grows like m^2 hence $mIC \sim A + C'$ for some $m > 0$ and some effective divisor C'. In particular, the classes of both A and C' are in $\overline{NE}(X)$.

Therefore general ample divisor A lies in same ray generated by C, on the other hand, ample divisors can form a basis of $N^1(X)_{\mathbb{R}}$ if general ample divisor lies in the ray, the basis of $N^1(X)_{\mathbb{R}}$ lies in it.

This implies that the class of C cannot generate any extremal ray unless $\overline{NE}(X)$ is just a half-line and $\rho(X) = 1$. □

5 Examples

In the final part of today's lecture, we provide some examples that Mori cone can be computed explicitly.

5.1 Smooth projective curves

We claim the Picard group of smooth projective curve is direct sum of the Jacobian of the curve with **Z**.

Proof. Conisder the following exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to 0$$

since for smooth projective curve $H^2(X, \mathbb{Z})$ is free and isomorphic to $\mathbb{Z}$ therefore the sequence of $\mathbb{Z}$ module split, that is

$$\mathrm{Pic}(X) = \mathrm{Pic}^0(X) \oplus \mathbb{Z}$$

as a consequence the Picard number being 1, and the Mori cone is a half line. □

5.2 Projective bundles

Projective bundle is standard model for Mori-fibre space(fiber type contraction)

We have the following projective bundle formula

Theorem 11. *Let Y be a smooth projective variety over $\mathbb{C}$, and $\mathcal{E}$ be a locally free sheaf on X, it will induce a projective bundle $f : \mathbb{P}(\mathcal{E}) \to Y$. Then any divisor $D \in Div(\mathbb{P}(\mathcal{E}))$ has the form*

$$D \sim f^*G + mH$$

for some $G \in Div(Y)$ and H be the hyperplane divisor on $\mathbb{P}(\mathcal{E})$. In particular $Pic(\mathbb{P}(\mathcal{E})) = Pic(Y) \oplus \mathbb{Z}$.

By our previous discussion if two curves C, C' lies in the fibers, the interestion number

$$C \cdot D = mC\dot{H},\ C' \cdot D = mC' \cdot$$

therefore the ratio between $C \cdot D$ and $C' \cdot D$ does not change when D varies. Therefore they lies in the same numerical class. That is $\text{NE}(f)$ is a extreme ray in $\text{NE}(X)$ if we take closure in $N_1(X)_{\mathbb{R}}$, the extreme ray still is itself, therefore $\text{NE}(f) = \overline{\text{NE}(f)}$ which is now a extreme ray of the Mori cone $\overline{\text{NE}(X)}$, therefore the projective bundle always has a extreme ray coming from the fiber that it contracted.

In the rest of this section, we will list some concrete examples of projective bundles.

(1)$\mathbb{P}^1$ bundle over the rational curve $\mathbb{P}^1$ (Hirzebruch surface)

Recall that $\mathbb{P}^1$ bundles coming from projectivization of rank 2 vector bundle. In this case the Picard number is 2, by Grotendieck classification theorem for vector bundles, it has the form $\mathcal{O}(a) \oplus \mathcal{O}(b)$ we can normalize it so that the $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ always has the form $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$ which is by definition the Hirzebruch surface $\mathbb{F}_n$(since the fiber of Hirzebruch surface are rational curves this is a standard example of Mori fibre space). By the theory of algebraic surface there is a section e on $\mathbb{F}_n$ with self-intersection number $-n$.

In this case the Mori cone has two extreme ray, one generated by the rays contained in the fibers, another one is generated by e.

Proof. Since it has Picard number 2 by theorem 9, we claim the cone of curves $\text{NE}(X)$ is already a closed cone.

By theorem 6, the ray generated by curves in the fibers is extreme for $\text{NE}(X)$. Since $e^2 = -n$ by theorem it's extreme ray for the Mori cone, therefore in particular it will extreme for $\text{NE}(X)$. They are not the same extreme ray for $e \cdot C = 1$ and $e^2 = -n$ for the curve C lies in the fibers.

□

(2)Projective bundle over a curve with higher genus(Mumford example)

There is a curve C of genus at least 2 and a locally free sheaf $\mathcal{E}$ of rank 2 on C such that the divisor D corresponding to the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is strictly nef (i.e. positively intersects every curve on $X = \mathbb{P}(\mathcal{E})$) but not ample.

As we have shown, D is numerically trivial on some extremal face of $\overline{NE}(X)$. (that is)

Since $\rho(X) = 2$ and since D is not numerically trivial, $\overline{NE}(X)$ has exactly two extremal rays and D is trivial on one of them, say R_1. Therefore, since D is strictly nef, R_1 does not contain the class of any curve on X. The other extremal ray R_2 is generated by the curves in the fibres of $X \to C$.

5.3 Blowing ups

Blowing ups is standard model for divisorial contraction.

5.4 Blowing up points on $\mathbb{P}^2$

Blowing up points on $\mathbb{P}^2$ produce many interesting examples in birational geometry.

Blowing up 1 point on $\mathbb{P}^2$

Blowing up 1 point on $\mathbb{P}^2$ has a projective bundle structure. The picture below shows the geometric intuition:

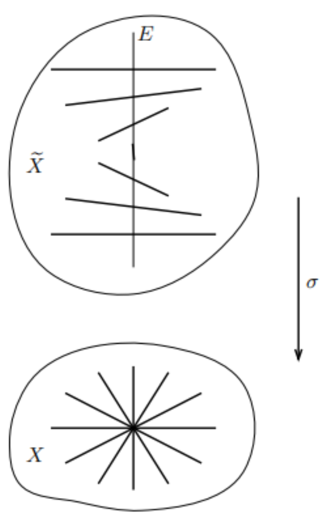

Figure 1:

We can notice from the picture that it has a projective bundle structure by contracting the strict transform of the lines passing through p on $\mathrm{Bl}_p(\mathbb{P}^2)$.

And as we have discussed in above the fibers of the projective bundle provides an extreme ray of the Mori cone, on the other hand the exceptional divisor contracted by the blowing up provides another extreme ray for $\mathrm{NE}(X)$ since taking closure does not change the ray, it's also the extreme ray for $\overline{NE(X)}$.

Blowing up 6 points on $\mathbb{P}^2$ that not lies on a conic

Blowing up 6 points that not lies on a conic are correspond to the cubic surface in $\mathbb{P}^2$ (which is therefore Fano surface, we will see later in lecture about Cone theorem that for Fano variety the extreme faces of Mori cone are finite).

Indeed in this example it has 27 lines on it, all of them are (-1) curves, it has 7 extreme face.

On interesting approach to counting the number of lines on the cubic surface is something called Schubert calculus.

More explicit calculations show that $N_1(X) \simeq \mathbb{R}^7$ and that $\overline{NE}(X)$ has no more extremal rays.

Blowing up 9 points on $\mathbb{P}^2$(Mukai example)

Finally, there are surfaces X which have infinitely many -1 -curves. So, they have infinitely many extreml rays. An example of such a surface is the blow up of the projective plane at nine points which are the base points of a general pencil of cubics.