Extension of MMP Fall 2025

Note III.6 — 11, 14, 2025 (draft version round 1)

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The aim of this note is to present the technique of extension of mmp, with varies applications of this technique. We will prove the following different version of extension theorems.

Theorem 1 (1st version extension theorem)**.** Let $f : X \to S$ be a flat, proper morphism of normal complex analytic varieties. Assume that X_0 is projective with canonical singularities. Then every sequence of steps of K_{X_0}-mmp

$$X_0 \dashrightarrow X_0^1 \dashrightarrow \cdots \dashrightarrow \cdots ,$$

extends to a sequence of steps of K_X-mmp relative over $U \subset S$

$$X/U \dashrightarrow X^1/U \dashrightarrow \cdots \dashrightarrow \cdots .$$

Theorem 2 (2nd version extension theorem)**.** Let $g : X \to S$ be a flat, proper morphism with connected fibers from a generalized Kähler pair $(X, B + \boldsymbol{\beta})$ to a smooth, connected, and relatively compact curve, such that the support of the boundary divisor B does not contain the central fiber X_0. Assume that $(X_0, B_0 + \boldsymbol{\beta}_0)$ is $\mathbb{Q}$-factorial, projective, with canonical singularities, where

$$(K_X + X_0 + B + \boldsymbol{\beta}_X)|_{X_0} = K_{X_0} + B_0 + \boldsymbol{\beta}_0,$$

and the negative part of divisorial Zariski decomposition satisfies the relation

$$N(K_{X_0} + B_0 + \boldsymbol{\beta}_0) \wedge B_0 = 0.$$

Then every sequence of transcendental $(K_{X_0} + B_0 + \boldsymbol{\beta}_0)$-mmp steps

$$(X_0, B_0 + \boldsymbol{\beta}_0) \dashrightarrow (X_0^{(1)}, B_0^{(1)} + \boldsymbol{\beta}_0^{(1)}) \dashrightarrow (X_0^{(2)}, B_0^{(2)} + \boldsymbol{\beta}_0^{(2)}) \dashrightarrow \ldots$$

extends to a sequence of $(K_X + B + \boldsymbol{\beta}_X)$-negative proper meromorphic maps

$$(X, B + \boldsymbol{\beta})/U \dashrightarrow (X^{(1)}, B^{(1)} + \boldsymbol{\beta}^{(1)})/U \dashrightarrow (X^{(2)}, B^{(2)} + \boldsymbol{\beta}^{(2)})/U \dashrightarrow \ldots ,$$

over some open neighborhood $U \subset S$ of 0.

The major references are [Kol21], [HLR25] and [LLR26].

Contents

[1]**WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

1 Kollár's extension of mmp technique

Theorem 3 ([Kol21, Theorem 2]). Let $f : X \to S$ be a flat, proper morphism of normal complex analytic varieties. Assume that X_0 is projective with canonical singularities. Then every sequence of steps of K_{X_0}-mmp

$$X_0 \dashrightarrow X_0^1 \dashrightarrow \cdots \dashrightarrow \cdots ,$$

extends to a sequence of steps of K_X-mmp relative over $U \subset S$

$$X \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow \cdots .$$

PROOF IDEA 4. There are two essential ingredients appear in the proof.

(1) First, we need the extend the analytic contractions. More details about the 1st part can be found in Note-3 Analytic Contractibility Theorem: Section 3 Kollár–Mori's extension theorem). The key point is to show that obstruction of the extension vanishes when $R^1 f_* \mathcal{O}_X = 0$.

(2) Second, we need to prove the existence of flip after extension. To prove this, assume we extend the flipping contraction to $f : X \to Z$, we then try to show that

$$\bigoplus_{k \geq 0} \omega_Z^{[k]}$$

is finite generated. Then the flip exists and equal to Proj $\bigoplus_{k \geq 0} \omega_Z^{[k]}$. Since

$$\mathrm{Proj}(\bigoplus_{k \geq 0} \omega_Z^{[k]}) = \mathrm{Proj}(\bigoplus_{k \geq 0} \omega_Z^{[k]}(kZ_0)).$$

It's sufficient to prove the finite generation of

$$\bigoplus_{k \geq 0} \omega_Z^{[k]}(kZ_0).$$

Assume for simplicity that Z is smooth and Z_0 is a smooth hypersurface divisor (otherwise we may need to take a embeded resolution, see proof below), we then consider the following diagram

$$\begin{array}{ccc} \bigoplus_{k \geq 0} S^k(\omega_Z(Z_0)) & \xrightarrow{\Phi} & \bigoplus_{k \geq 0} \omega_Z^k(kZ_0) \\ \downarrow{\scriptstyle S^* r} & & \downarrow{\scriptstyle r} \\ \bigoplus_{k \geq 0} S^k(\omega_{Z_0}) & \xrightarrow{\Phi_0} & \bigoplus_{k \geq 0} \omega_{Z_0}^k \end{array}$$

where S^k means the symmetric power of the coherent sheaves, and $r : \omega_Z^k(kZ_0) \to \omega_{Z_0}^k$ is the standard restriction map, and by the adjunction formula we have

$$\omega_Z(Z_0) \otimes \mathcal{O}_{Z_0} \cong \omega_{Z_0}, \ \omega_Z^k(kZ_0) \otimes \mathcal{O}_{Z_0} \cong \omega_{Z_0}^k$$

By Nakayama's extension theorem, r is surjective map. Since symmetric power preserves surjectivity, S^*r is surjective.

Since the flip exists on the central fiber, the $\bigoplus_{k\geq 0}\omega_{Z_0}^k$ is finite generated. Therefore Φ_0 is surjective. Hence Φ is surjective as well. Since

$$\mathrm{Sym}^\bullet(\omega_Z(Z_0)) = \bigoplus_{k\geq 0} S^k(\omega_Z(Z_0)),$$

is finite generated (by degree 1 elements), $\bigoplus_{k\geq 0}\omega_Z^k(kZ_0)$ is finite generated.

We now give a complete proof of the result.

Proof. □

Theorem 5 (Nakayama's Extension Theorem)**.** Let $\pi : Y \to S$ be a projective, bimeromorphic morphism of analytic spaces, Y smooth and S normal. Let $D \subset Y$ be a smooth, non-exceptional divisor. Then the restriction map $\pi_*\omega_Y^m(mD) \to \pi_*\omega_D^m$ is surjective for $m \geq 1$.

2 Generalization of the result to the generalized Kähler pairs

Recently, we make the following generalization of the result.

Theorem 6 ([HLR25, Theorem 4.15])**.** Let $g : X \to S$ be a flat, proper morphism with connected fibers from a generalized Kähler pair $(X, B + \boldsymbol{\beta})$ to a smooth, connected, and relatively compact curve, such that the support of the boundary divisor B does not contain the central fiber X_0. Assume that $(X_0, B_0 + \boldsymbol{\beta}_0)$ is $\mathbb{Q}$-factorial, projective, with canonical singularities, where

$$(K_X + X_0 + B + \boldsymbol{\beta}_X)|_{X_0} = K_{X_0} + B_0 + \boldsymbol{\beta}_0,$$

and the negative part of divisorial Zariski decomposition satisfies the relation

$$N(K_{X_0} + B_0 + \boldsymbol{\beta}_0) \wedge B_0 = 0.$$

Then every sequence of transcendental $(K_{X_0} + B_0 + \boldsymbol{\beta}_0)$-mmp steps

$$(X_0, B_0 + \boldsymbol{\beta}_0) \dashrightarrow (X_0^{(1)}, B_0^{(1)} + \boldsymbol{\beta}_0^{(1)}) \dashrightarrow (X_0^{(2)}, B_0^{(2)} + \boldsymbol{\beta}_0^{(2)}) \dashrightarrow \dots$$

extends to a sequence of $(K_X + B + \boldsymbol{\beta}_X)$-negative proper meromorphic maps

$$(X, B + \boldsymbol{\beta})/U \dashrightarrow (X^{(1)}, B^{(1)} + \boldsymbol{\beta}^{(1)})/U \dashrightarrow (X^{(2)}, B^{(2)} + \boldsymbol{\beta}^{(2)})/U \dashrightarrow \dots,$$

over some open neighborhood $U \subset S$ of 0.

PROOF IDEA 7. Let us highlight the idea of the proof. We will divide the proof into 4 steps:

(a) **Step 0. We use the analytic contraction theorem** of Kollár–Mori to extend the contraction from the central fiber X_0 to some neighborhood (as in the proof of Theorem 3).

(b) **Step 1. We try to show the flips extend to flips.** The idea is very similar to the proof of [DH24a, Theorem 5.12]. We try to construct the log canonical model locally. Then using uniqueness of log canonical model (up to numerical equivalence), we can glue them together to global flips.

The major difficulty then is how to replace the generalized pair as log pair. To do this, one need to prove that the extension admits rational singularity, and then the result follows from [KM92, Theorem 12.1.1].

(c) **Step 2. Prove the flip restrict to flip on the central fiber**, that is we need to show that the restriction of the flip on the central fiber does not contract and extract divisors. We will prove it by contradiction using monotonicity lemma and canonical singularity assumption.

(d) **Step 3. We Deal with the extension of the divisorial contraction.** Assume that on the central fiber, we have a $K_{X_0} + B_0 + \boldsymbol{\beta}_0$-divisorial contraction $X_0 \to Z_0$. We then extend it to $X \to Z$. The problem is to show that $(Z, B_Z + \boldsymbol{\beta}_Z)$ is still a generalized pair. To prove this, we first run a relative $(K_X + B + \boldsymbol{\beta})$ canonical model over Z

$$X \dashrightarrow Z^c/Z,$$

so that $(Z^c, B_{Z^c} + \boldsymbol{\beta}_{Z^c})$ is a generalized pair. We then show that in a neighborhood of Z_0, $Z^c \to Z$ is an isomorphism, hence $(Z, B_Z + \boldsymbol{\beta}_Z)$ is a generalized pair near Z_0.

Proof. □

3 Base point freeenss and termination of the extension

Note that the extension of mmp technique tells us nothing about the termination. We expect that when the mmp on the central fiber terminates, then the mmp for nearby fibers will end with nef adjoint class as well. To achieve this, we typically require an additional bigness assumption for the adjoint class on the central fiber.

Theorem 8 ([DFH11, Lemma 3.9]). Let $f : X \to S$ be a flat projective morphism from klt pai (X, B) to a smooth curve S, with $d = \dim_{\mathbb{C}}(X/S)$. Fix a point $0 \in S$ and assume that X_0 is a normal variety that not contained in the support of B.

Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $L|_{X_0}$ is nef and big and $aL|_{X_0} - (K_X + B)|_{X_0}$ is nef and big for some $a \geq 0$. Then $L|_{X_t}$ is nef and big for any t in a neighborhood of $0 \in S$.

PROOF IDEA 9. We may denote the line bundle M as L or $aL - (K_X + B)$ and treat them uniformly.

Step 1. (Prove $M|_{X_t}$ on very general fibers are nef). It follows by the standard cycle space argument.

Step 2. (Prove $M|_{X_t}$ on very general fibers are big). Using the fact that $M|_{X_0}$ is big and nef, we know that $h^0(M|_{X_0})$ growth maximally, while $h^i(M|_{X_0}) \leq Ck^{d-1}$. Combined with upper semi-continuity of cohomological dimension and constancy of the Euler characteristics, the $h^0(M|_{X_t})$ on the very general fibers growth maximally as well. And we denote the very general locus W.

Step 3. (Prove openness of big locus using relative Kodaira map). By generic surjectivity of $H^0(\mathcal{O}_X(mL)) \to H^0(\mathcal{O}_{X_t}(mL))$, after shrinking W, $\phi_{|mL|_{X_t}|} = \phi_{|mL|}\big|_{X_t}$ for all $t \in W$. Since bigness of $L|_{X_t}$ for $t \in W$, the Kodaira map $\phi_{|mL|}|_{X_t}$ are generic finite for $t \in W$. Using openness of quasi-finite locus this implies $\phi_{|mL|}|_{X_t}$ are generic finite for $t \in U$ in some dense open subset.

Step 4. (Prove openness of the nef locus). The idea is to show the non-nef locus Σ satisfies the inclusion relation

$$\Sigma \subset f(\mathrm{Bs}(|mL|)) \subsetneq S.$$

Since $f(\mathrm{Bs}(|mL|))$ is a proper Zariski closed subset, this force Σ to be finite union of points in the curve S.

Proof. □

We have the following generalization to the Kähler pair.

Theorem 10 ([HLR25])**.** Let $f : X \to S$ be a proper surjective morphism from a normal $\mathbb{Q}$-factorial generalized Kähler pair $(X, B + \boldsymbol{\beta})$ onto a smooth, connected, relatively compact curve S, and ω a Kähler form on X.

Assume that the restriction to the central fiber $(X_0, B_0 + \boldsymbol{\beta}_0)$ is a projective generalized klt pair, such that $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is nef and big. Then $K_X + B + \boldsymbol{\beta}_X$ is nef and big over U for some open neighborhood $U \subset S$.

Remark 11. The theorem above can be viewed as a generalization of Theorem 8. If we assume that $L = K_X + B$ in theorem 8.

PROOF IDEA 12. We assume that $(X_0, B_0 + \boldsymbol{\beta}_0)$ is a projective gklt pair and $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is big and nef. By trascendental base point freeness for projective generalized pair [DH24b], there exists a bimeromorphic contraction $f_0 : X_0 \to Z_0$ and a Kähler form ω_{Z_0} such that

$$K_{X_0} + B_0 + \boldsymbol{\beta}_0 = f_0^*(\omega_{Z_0}).$$

By Kawamata–Viehweg vanishing theorem, we have $R^1(f_0)_*(\mathcal{O}_{X_0}) = 0$, and therefore one can extend f_0 to a bimeromorphic contraction $f : X \to Z$ over some neighborhood of 0. Since $K_{X_0} + B_0 + \boldsymbol{\beta}_0 \equiv_{Z_0} 0$ on the central fiber, it's sufficient to prove that $K_X + B + \boldsymbol{\beta}_X \equiv_Z 0$, and then by Lemma 13, we know that

$$K_X + B + \boldsymbol{\beta}_X = f^*\omega$$

for some $\omega \in H^{1,1}_{\mathrm{BC}}(Z, \mathbb{R})$. Since ω_0 is Kähler, this implies that $K_X + B + \boldsymbol{\beta}_X$ is nef and big over U.

Proof. □

Lemma 13. Let $f : X \to Y$ be a proper morphism with connected fibers between normal compact complex spaces with rational singularities. Assume that one of the following two conditions hold:
(1) X and Y are in Fujiki's class $\mathcal{C}$ and f is bimeromorphic, or,
(2) there is an effective $\mathbb{Q}$-divisor $B \geq 0$ such that (X, B) is klt, $-(K_X + B)$ is f-nef-big and f is projective.

Pull back $f^*: H^{1,1}_{\mathrm{BC}}(Y) = H^1(Y, \mathcal{H}_Y) \to H^{1,1}_{\mathrm{BC}}(X) = H^1(X, \mathcal{H}_X)$ and a $f^*: H^2(Y, \mathbb{R}) \to H^2(X, \mathbb{R})$ are both injective, with

$$\mathrm{Im}\,(f^*) = \Big\{\alpha \in H^{1,1}_{\mathrm{BC}}(X) \mid \alpha \cdot C = 0 \ \text{ for all curves } C \subset X \ s.t. \ \ f_*(C) = \mathrm{pt}\}$$

and

$$\mathrm{Im}\,(f^*) = \{\alpha \in H^2(X) \mid \alpha \cdot C = 0 \ \text{ for all curves } C \subset X \ s.t. \ \ f_*(C) = \mathrm{pt}\}$$

As an immediate corollary, we can prove the desired "termination" for the generalized Kähler pairs.

Corollary 14. Let $f : X \to S$ be a flat, proper morphism with connected fibers from a generalized Kähler pair $(X, B + \boldsymbol{\beta})$ to a smooth connected and relatively compact curve, such that the support of the boundary divisor B does not contain the central fiber X_0.

Assume that there is a projective generalized gklt pair $(X_0, B_0 + \boldsymbol{\beta}_0)$, such that $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is big, and there is a transcendental $(K_{X_0} + B_0 + \boldsymbol{\beta}_0)$-mmp terminates with some minimal model

$$(X_0, B_0 + \boldsymbol{\beta}_0) \dashrightarrow (X_0^{(1)}, B_0^{(1)} + \boldsymbol{\beta}_0^{(1)}) \dashrightarrow (X_0^{(2)}, B_0^{(2)} + \boldsymbol{\beta}_0^{(2)}) \dashrightarrow \cdots \dashrightarrow (X_0', B_0' + \boldsymbol{\beta}_0').$$

Then it extends to a sequence of $(K_X + B + \boldsymbol{\beta})$-negative contractions

$$(X, B + \boldsymbol{\beta})/U \dashrightarrow (X^{(1)}, B^{(1)} + \boldsymbol{\beta}_0^{(1)})/U \dashrightarrow (X^{(2)}, B^{(2)} + \boldsymbol{\beta}^{(2)})/U \dashrightarrow \cdots \dashrightarrow (X', B' + \boldsymbol{\beta}')/U.$$

such that $K_{X_t'} + B_t' + \boldsymbol{\beta}_t'$ is nef for any $t \in U \subset S$.

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