Hacon-Mckernan's Extension Theorem Notes	Summer 2024
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	Yi Li

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1 Overview

The aim of this note is try to explain the Hacon-Mckernan's extension theorem and present varies applications of this powerful tool. The theorem can be stated as follows.

Theorem 1.1. Let $\pi : X \to Z$ be a projective morphism (from a smooth quasi-projective variety) to a normal affine variety.

Let $(X, D = S + A + B)$ be a PLT pair of dimension n where X and S are smooth, $D \in \mathrm{Div}_{\mathbb{Q}}(X), \llcorner D \lrcorner = S, A$ is a general ample $\mathbb{Q}$-divisor,

1. $(S, \Omega = (D - S)|_S)$ is canonical (terminal), and

2. the stable base locus
$$\mathrm{supp}\, S \not\subset \mathbf{B}(K_X + D)$$

   (3) For any sufficiently divisible $m > 0$, let

$$F_m = \mathrm{Fix}\left(\left|m\left(K_X + D\right)\right|_S\right)/m \text{ and } F = \lim F_{m!}$$

If $0 < \epsilon \in \mathbb{Q}$ is such that $\epsilon(K_X + D) + A$ is ample, $\Phi \in \operatorname{Div}_{\mathbb{Q}}(S)$ and $0 < k \in \mathbb{Z}$ such that kD and $k\Phi$ are Cartier, and

$$\Omega \wedge \lambda F \leq \Phi \leq \Omega$$

where $\lambda = 1 - \epsilon/k$,

Then

$$|k(K_S + \Omega - \Phi)| + k\Phi \subset |k(K_X + D)|_S.$$

Let us briefly sketch the idea in the proof of the theorem. We will first prove a technical lemma which allows one to lift section after ample twist (which also appears in the algebraic proof of invariance of plurigenera). To prove this lemma, we first observe that by the Serre vanishing theorem the restricted linear series coinside with the linear series of the restriction divisor when twisted with sufficient ample divisor, .

In the second step, we will going to prove the main theorem of this note.

The major references will be [HM10] and [**HaconKovacsBook**]

2 Interpretation of Hacon-Mckernan's extension theorem

The goal of this section to understand the meaning of Hacon-Mckernan's extension theorem.

Remark 2.1 (Limit of the sequence of fixed divisors exist).

Remark 2.2 (Obstruction divisor is necessary). We call the divisor $\Xi := \Omega \wedge \lambda F$ the obstruction divisor. The following simple example shows that without the obstruction divisor, the result needs not to be true.

Remark 2.3 (Few remarks about the theorem).

1. The first thing is how to understand this theorem? Things become more clear if we can choose $\Phi = 0$. The result says that $|k(K_S + \Omega)| = |k(K_X + D)|_S$, which is what we hope, lifting pluricanonical section. However, we can not expect the result to as nice as we hope. Indeed we have the following example: Let X be the blow-up of $\mathbb{P}^2$ at a point q, E the exceptional divisor, S the strict transform of a line through q and L_i the strict transforms of general lines on $\mathbb{P}^2$. We let $\Delta = S + \frac{2}{3}(L_1 + L_2 + L_3)$ so that $\Omega = \frac{2}{3}(E + L_1 + L_2 + L_3)|_S = \frac{2}{3}(p + p_1 + p_2 + p_3)$. For $k = 3l$, we have that $|k(K_S + \Omega)| = |2lp| = \mathbb{P}^{2l}$ whereas $|k(K_X + \Delta)| = |2lE|$. Thus we need some modification on the result. The idea is ...

2. The second thing is to show that assumptions in the theorem is essential.

3. We get divisor F via a limiting process. Since $(m + 1)! = (m + 1)m!$, we have $F_{(m+1)!} \leq F_{m!}$ and so $\lim F_{m!} = \inf\{F_m\}$ exists.

4. About the analytic proof. The analytic proof is much more stronger. This can be seen already from the Ohsawa-Takegoshi extension theorem, which only requires some positivity weaker than ampleness. If you want to prove some result without ampleness condition, the analytic method may be necessary, as a paid-off the proof will sometimes be more complicated.

3 Proof of Hacon-Mckernan's extension theorem

3.1 Lifting sections that vanish along the multiplier ideal

By the standard arguement of Nadel vanishing theorem, we can prove the following lemma. The lemma shows that

Lemma 3.1 (Lifting sections that vanish along the multiplier ideal)**.** Let X be a smooth quasi-projective variety, Δ be a reduced divisor with simple normal crossings support and $D \geq 0$ be a $\mathbb{Q}$-Cartier divisor whose support contains no non-klt center of (X, Δ). Let $\pi : X \to Z$ be a projective morphism to a normal affine variety Z and N be a Cartier divisor on X such that $N - D$ is ample. Then

$$H^i\left(X, \mathscr{J}_{\Delta, D}\left(K_X + \Delta + N\right)\right) = 0 \quad \text{for } i > 0,$$

and if S is a component of Δ, then the homomorphism

$$H^0\left(X, \mathscr{J}_{\Delta, D}\left(K_X + \Delta + N\right)\right) \twoheadrightarrow H^0\left(S, \mathscr{J}_{(\Delta - S)|s, D|_S}\left(K_X + \Delta + N\right)\right)$$

is surjective.

The proof is not hard

Proof. □

3.2 Lifting section with ample twist

As have been mentioned in Section 1. The first step of proving the Hacon Mckernan extension theorem is the following lemma which allows to lift section after sufficient ample twist.

Lemma 3.2 (Lifting section with ample twist)**.** Let $\pi : X \to Y$ be a projective morphism from a smooth quasi-projective variety to an affine variety, $\Delta = \sum \delta_i \Delta_i$ be a divisor with simple normal crossings support and rational coefficients $0 \leq \delta_i \leq 1$ and S an irreducible component of $\lfloor \Delta \rfloor$. Let $k > 0$ be an integer such that $D = k\left(K_X + \Delta\right)$ is Cartier.

If $\mathbf{B}(D)$ contains no non-klt centers of $(X, \lceil \Delta \rceil)$ and if A is a sufficiently ample Cartier divisor, then for all $m > 0$,

$$\mathscr{J}_{\|mD|S\|} \subseteq \mathscr{J}_{(\lceil \Delta \rceil - S)|s, \|mD + A\|_S} \quad,$$
$$\pi_* \mathscr{J}_{\|mD|_S\|}(mD + A) \subseteq \mathrm{Im}\left(\pi_* \mathscr{O}_X(mD + A) \to \pi_* \mathscr{O}_S(mD + A)\right)$$

Let us make few remarks about the lemma:

Remark 3.3. 1. The result of the lemma above says that

Proof. □

4 Applications of Hacon Mckernan extension theorem

In this section, we will summarize all the known applications of Hacon-Mckernan type extension theorem.

4.1 In the existence of KLT flips theorem

4.2 In the Shokurov rational connectedness conjeture

4.3 In the deformation of Mori chamber for a family of Fano varieties

4.4 In the Boundedness of general type problem

References

[HM10] C.D. Hacon and J. McKernan. "Existence of minimal models for varieties of log general type. II". In: *J. Amer. Math. Soc.* 23.2 (2010), pp. 469–490.