deFernex-Hacon extension Readings Notes Fall 2025

Note I.3 — 2025 09 12 (draft version 0)

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1 Overview

The aim of this note is to try to prove the deFernex-Hacon extension theorem [DFH11] and gives varies applications.

Theorem 1.1 ([DFH11, Theorem 4.6])**.** Let $f : X \to T$ be a flat projective morphism of normal varieties where T is an affine curve.

Assume that for some the fiber is normal and $\mathbb{Q}$-factorial, D is an effective $\mathbb{Q}$-divisor whose support does not contain X_0 such that $\left(X_0, D|_{X_0}\right)$ is a Kawamata log terminal pair with canonical singularities. Assume that $N^1(X/T) \to N^1(X_0)$ is surjective, and either $D|_{X_0}$ or $K_{X_0} + D|_{X_0}$ is big, and that one of the following two conditions is satisfied:
(a) $\mathrm{SBs}\,(K_X + D)$ does not contain any component of $\mathrm{Supp}\left(D|_{X_0}\right)$, or
(b) $D|_{X_0} - aK_{X_0}$ is ample for some $a > -1$.

Let L be an integral Weil $\mathbb{Q}$-Cartier divisor whose support of L does not contain X_0 and such that $L|_{X_0} \equiv k\left(K_X + D\right)\big|_{X_0}$ for some rational number $k > 1$. Then the restriction map

$$H^0\left(X, \mathcal{O}_X(L)\right) \to H^0\left(X_0, \mathcal{O}_{X_0}\left(L|_{X_0}\right)\right)$$

is surjective.

2 deFernex-Hacon extension theorem

To deFernex-Hacon extension theorem. We will divide the proof into two steps, first we try to show under certain condition, the relative MMP restrict to the MMP on the central fiber. In the second step, we use some MMP argument proving the deFernex-Hacon extension theorem.

2.1 Restriction of relative MMP on the central fiber

Let us first study the restriction of relative MMP on the central fiber.

Theorem 2.1 ([DFH11, Theorem 4.1])**.** Let $f : X \to T$ be a flat projective morphism of normal varieties where T is an affine curve. Assume that

(a) for some $0 \in T$ the fiber X_0 is normal and $\mathbb{Q}$-factorial,
(b) $N^1(X/T) \to N^1(X_0)$ is surjective
(c) D is an effective $\mathbb{Q}$-divisor whose support does not contain X_0 such that
(d) $\left(X_0, D|_{X_0}\right)$ is a KLT pair with canonical singularities.

Let $\psi : X \to Z$ be the contraction over T of a $(K_X + D)$-negative extremal ray of $\overline{\mathrm{NE}}(X/T)$ (it's relative contraction, I mean contraction on the fiber direction), and let $Z_0 = \psi(X_0)$.

If $\psi_0 := \psi|_{X_0} : X_0 \to Z_0$ is not an isomorphism, then it is the contraction of a $\left(K_{X_0} + D|_{X_0}\right)$-negative extremal ray, and

(Case 1) If ψ is of fiber type, then so is ψ_0.

(Case 2) If ψ is a divisorial contraction of a divisor G, then ψ_0 is a divisorial contraction of $G|_{X_0}$ (in particular $G|_{X_0}$ is irreducible), and $N^1(Z/T) \to N^1(Z_0)$ is surjective.

(Case 3) Assume additionally that either (a) $\mathrm{SBs}(K_X + D/Z)$ does not contain any component of $\mathrm{Supp}\left(D|_{X_0}\right)$, or (b) $D|_{X_0} - aK_{X_0}$ is nef over Z_0 for some $a > -1$.

If ψ is a flipping contraction and $\psi^+ : X^+ \to Z$ is the flip, then ψ_0 is a flipping contraction and, denoting $X_0^+$the proper transform of X_0 on X^+, the induced morphism $\psi_0^+ : X_0^+ \to Z_0$ is the flip of $\psi_0 : X_0 \to Z_0$. Moreover $N^1(X^+/T) \to N^1\left(X_0^+\right)$is surjective.

PROOF IDEA 2.2. The proof is a bit involved. We divisor the proof into several steps.

Step 1. By cone theorem. There is an ample $\mathbb{Q}$-divisor H on X such that the contraction ψ is defined by $|m(K_X + D + H)|$ for any sufficiently divisible $m \geqq 1$. We try to show that the restriction of the contraction is induced by the linear system $|m(K_{X_0} + D_0 + H_0)|$ by showing the surjectivity of

$$H^0(\mathcal{O}_X(m(K_X + D + H))) \to H^0(\mathcal{O}_{X_0}(m(K_{X_0} + D_0 + H_0))).$$

In particular, it tells us that the restriction is a $K_{X_0} + D_0$-negative extremal contraction.

Step 2. We consider the restriction of the divisorial contraction case, we try to show that the (prime) divisor G being contracted by $\psi : X \to Z$ has the restriction $G_0 = G|_{X_0}$ such that all the components being contracted by ψ_0 and it's prime divisor on X_0.

Step 3. We consider the flipping contraction case by contradiction. Assume that the restriction is a divisorial contraction, then we try to prove the $\mathbb{Q}$-Cartier of $K_Z + \psi_* D$ (or K_Z). However, this is impossible for flipping contraction.

Step 4. We try to prove the flip restrict to flip on the central fiber. We first show that the restriction of flip on X_0 is isomorphism in codimension 1 and then show that X_0^+ is actually the canonical model of X_0 over Z_0. In particular, $X_0 \dashrightarrow X_0^+$ is a flip over Z_0.

Before proving the theorem, let us first have a look at the surjectivity assumption on specialization of $N^1(X/T)$. This condition are satisfied in varies situation.

Proposition 2.3 (Specialization of $N^1(X/T)$ is surjective for Fano type Fibration)**.**

2.2 deFernex Hacon extension theorem

Theorem 2.4 ([DFH11, Theorem 4.6])**.** Let $f : X \to T$ be a flat projective morphism of normal varieties where T is an affine curve.

Assume that for some the fiber is normal and $\mathbb{Q}$-factorial, D is an effective $\mathbb{Q}$-divisor whose support does not contain X_0 such that $\left(X_0, D|_{X_0}\right)$ is a Kawamata log terminal pair with canonical singularities. Assume that $N^1(X/T) \to N^1\left(X_0\right)$ is surjective, and either $D|_{X_0}$ or $K_{X_0} + D|_{X_0}$ is big, and that one of the following two conditions is satisfied:
(a) $\operatorname{SBs}\left(K_X + D\right)$ does not contain any component of $\operatorname{Supp}\left(D|_{X_0}\right)$, or
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Let L be an integral Weil $\mathbb{Q}$-Cartier divisor whose support of L does not contain X_0 and such that $L|_{X_0} \equiv k\left(K_X + D\right)\big|_{X_0}$ for some rational number $k > 1$. Then the restriction map

$$H^0\left(X, \mathcal{O}_X(L)\right) \to H^0\left(X_0, \mathcal{O}_{X_0}\left(L|_{X_0}\right)\right)$$

is surjective.

PROOF IDEA 2.5. The idea is very simple, we first run relative $(K_X + D)$-MMP over T. We then restrict the relative MMP on the central fiber and showing that it gives an absolute MMP on X_0. Hence we can further assume the nefness assumption in order to prove the extension theorem. Thus we can apply the Nadel vanishing to deduce the result. Note that the termination of the MMP requires the deformation openness of big and nef condition, which is possible thanks to the effective base point free theorem.

Proof. □

3 Applications of deFernex-Hacon extension theorem

3.1 Deformation of Nef Value

Using the deFernex-Hacon extension theorem we can prove the deformation invariance of nef value under additional assumptions.

Theorem 3.1.

Let us first reduce the theorem into the following proposition first.

Proposition 3.2.

3.2 Deformation of Moving cone

3.3 Rigidity of Toric Fano varieties

4 Totaro's example