Minimal model, good minimal model, canonical model in Family Fall 2025

Note I.4 — 2025 09 12 (in progress draft)

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1 Overview

The aim of this note is to study the behavior of minimal model, good minimal model and canonical models in family.

Contents

2 Deformation of minimal model, good minimal model and log canonial models

Given a log smooth family, the good minimal model behave well under deformation.

Theorem 2.1 ([HMX18, Theorem 1.2])**.** Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi : X \longrightarrow U$ be a projective morphism to a smooth variety U. Suppose that (X, Δ) is log smooth over U.

If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then (X, Δ) has a good minimal model over U and every fibre has a good minimal model.

PROOF IDEA 2.2.

3 Restriction of Flips and Canonical models on the Fibers

4 Restriction on non-klt locus

5 Abundance on the closure of the family

In this section, we will discuss the deformation closedness of abundance. We will only sketch the idea of the proof as what we did above. A complete proof can be found in

Theorem 5.1 ([HX13, Theorem 1.1])**.** Let $f : X \to U$ be a projective morphism of normal varieties, Δ a $\mathbb{Q}$-divisor such that (X, Δ) is a dlt pair and $S = \lfloor \Delta \rfloor$ the non-klt locus.

Assume that there exists an open subset $U^0 \subset U$ such that $\left(X^0, \Delta^0\right) := (X, \Delta) \times_U U^0$ has a good minimal model over U^0, and that any stratum of S intersects X^0.

Then (X, Δ) has a good minimal model over U.

PROOF IDEA 5.2. Let us briefly sketch the idea of the proof. The major idea is try to prove the finite generation of the log canonical ring $R(X/U, K_X + \Delta)$. And then the existence of good minimal model over U follows easily by the Proposition 5.3.

Step 1. We first find a canonical model on the Zariski open subset, then we try to compactify it on the closure.

$$
\begin{tikzcd}
 & X^o \arrow[rr, hook] \arrow[dl, dashed] \arrow[dd] & & X \arrow[dd] \\
\bar{Y}^o \arrow[rr, hook] \arrow[dr] & & \bar{Y} \arrow[dr] & \\
 & U^o \arrow[rr, hook] & & U.
\end{tikzcd}
$$

Step 2. We then take a birational modification on $\bar{Y}$ which keep the log canonical ring unchanged. As

$$Y \to \bar{Y}$$

such that $h_X : X \to Y$ factor through. The key point is under this process, the log canonical ring does not change.

$$h_{X*}\mathcal{O}_X\left(m\left(K_X + \Delta\right)\right) \cong \mathcal{O}_Y\left(m\left(K_Y + B + J\right)\right)$$

$$
\begin{tikzcd}
 & X^o \arrow[rr, hook] \arrow[dl, dashed] \arrow[dd] & & X \arrow[dd] \arrow[dr, "h_X"] & \\
\bar{Y}^o \arrow[rr, hook] \arrow[dr] & & \bar{Y} \arrow[dr] & & Y \arrow[ll] \\
 & U^o \arrow[rr, hook] & & U &
\end{tikzcd}
$$

Step 3. Prove existence of good minimal model $X \dashrightarrow Z/Y$. We then run the $K_X+\Delta$-mmp over Ys and find a relative good minimal model

$$X \dashrightarrow Z/Y.$$

Step 4. Run the $(K_Y+B+J+\epsilon(C+A))$-relative MMP over U, and lift it to Z.

We then prove the existence of $(K_Y+B+J+\epsilon(C+A))$-good MMP over U. And we can lift it onto $Z' \to Y'$ as the following diagram shows.

$$\begin{tikzcd}
Z \arrow[rr, dashed] \arrow[d] & & Z' \arrow[d, "h'"] \\
Y \arrow[rr, dashed] \arrow[dr] & & Y' \arrow[dl] \\
& U &
\end{tikzcd}$$

The point is under this process we have the following 3 condition holds.

(1) $(K_{Y'}+B'+J'+t\,(C'+A'))|_{Y'^0}$ is semi-ample over U^0 for $0 \le t \le \epsilon$.

(2) $(K_{Y'}+B'+J'+\epsilon\,(C'+A'))|_{Y'^0}$ is semi-ample over U.

(3) $K_X+\Delta+\epsilon h_X^*(A+C)$ has a minimal model over U, Z' equipped with a morphism $h' : Z' \to Y'$.

Step 5. Prove the termination of the mmp and mmp preserve the canonical ring. We then prove the termination of the mmp we constructed in step 4,

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] & & Z \arrow[rr, dashed] \arrow[dl, "h"] & & Z' \arrow[rr, dashed] \arrow[dl, "h'"] & & Z'_1 \arrow[r, dashed] \arrow[dl, "h'_1"] & \cdots \arrow[r, dashed] \arrow[dl] & Z'_N \arrow[dl, "h'_N"] \\
& Y \arrow[rr, dashed] \arrow[dr] & & Y' \arrow[rr, dashed] \arrow[dl] \arrow[dr] & & Y'_1 \arrow[r, dashed] \arrow[dl] \arrow[dr] & \cdots \arrow[r, dashed] & Y'_N \arrow[dl] & \\
& & U & & U & & U & &
\end{tikzcd}$$

On of the important point is the canonical ring does not change:

$$\boxed{R\,(X/U, K_X+\Delta) \cong R\left(Z'_N/U, K_{Z'_N}+\Delta_{Z'_N}\right)}$$

Step 6. Prove the existence of good minimal model on the closure using finite generation of canonical ring.

By Proposition 5.3, we can finish the proof using the finite generation of canonical ring

$$R(X/U, K_X+\Delta).$$

By Step 5, it's sufficient to prove the finite generation of $R\left(Z'_N/U, K_{Z'_N}+\Delta_{Z'_N}\right)$. By the nefness of $K_{Z'_N}+\Delta_{Z'_N}$, we can apply the the base point freeness result Theorem 5.4, so that reduce to the semi-ampleness of $K_{Z^0_N}+\Delta_{Z^0_N}$ over U, which follows by our assumption of the existence of good minimal model for $K_{X^0}+\Delta^0$ over U.

Proposition 5.3 (Finite generation criterion for existence of relative good minimal model, [HX13, Theorem 2.12])**.** Let $f : X \to U$ be a surjective projective morphism and (X, Δ) a dlt pair such that

(1) for a very general point $u \in U$, the fiber $\left(X_u, \Delta_u = \Delta|_{X_u}\right)$ has a good minimal model, and

(2) the ring $R\,(X/U; K_X + \Delta)$ is finitely generated.

Then (X, Δ) has a good minimal model over U.

Theorem 5.4 (Base point freeness in family, [HX13, Theorem 4.1])**.** Let $f : X \to U$ be a projective morphism and (X, Δ) a $\mathbb{Q}$-factorial dlt pair. Assume that there exists an open subset $U^0 \subset U$, such that

(1) the image of any strata S_I of $S = \lfloor \Delta \rfloor$ intersects U^0,

(2) $K_X + \Delta$ is nef and $(K_X + \Delta)|_{X^0}$ is semi-ample over U^0 where $X^0 = X \times_U U^0$, and

(3) for any component S_i of S, $(K_X + \Delta)|_{S_i}$ is semi-ample over U.

Then $K_X + \Delta$ is semi-ample over U.

References

[HMX18] Christopher D. Hacon, James McKernan, and Chenyang Xu. "Boundedness of moduli of varieties of general type". In: *J. Eur. Math. Soc. (JEMS)* 20.4 (2018), pp. 865–901. ISSN: 1435-9855,1435-9863. DOI: 10.4171/JEMS/778. URL: https://doi.org/10.4171/JEMS/778.

[HX13] Christopher D. Hacon and Chenyang Xu. "Existence of log canonical closures". In: *Invent. Math.* 192.1 (2013), pp. 161–195. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-012-0409-0. URL: https://doi.org/10.1007/s00222-012-0409-0.