Birational Geometry Reading Notes	Summer 2024
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1   Introduction

The aim of this note is to prove the theorem of Angehrn and Siu. The theorem gave a quadratic result for global generation of the adjoint bundle. The major difficulty in the proof of the theorem is how to construct an effective $\mathbb{Q}$-divisor D s.t. (a) the log canonical threshold $\mathrm{LCT}(D, x) = 1$, and (b) x is the isolated point the LC locus. Once we successfully construct such divisor D, the proof easily follows by appling the Nadal vanishing theorem to the exact sequence associated to the multiplier ideal sheaf $\mathcal{J}_D$.

Let us briefly sketch the idea on how to produce divisor D above: We start our construction by observing that the multiplier ideal will vanish at a given smooth point if the multiplicity of the divisor D_1 is sufficient large. We then inductively construct a sequence of divisors D_i such that the vanishing locus will strictly decrease

$$Z_1 = V(\mathcal{J}(D_1)) \supsetneq Z_2 = V(\mathcal{J}(D_2)) \supsetneq \ldots,$$

since the dimension strictly decrease, there is some divisor D_k such that $V(\mathcal{J}(D_k))$ is a point. The major problem that will appear in the induction process is that although $x \in X$ is smooth point of X but it can be singular point of Z_1. The idea of Angehrn and Siu is taking a curve passing through x such that all the nearby points on the curve are smooth points and therefore it's possible to construct some divisor $D_{2,t}$, then they use a limiting arguement and construct $D_{2,t} \to D_2$ and check that D_2 is the one that we hope.

This note is organized as follows: In Section 2. we will state the main theorem with some applications. In Section 3. we will summarize the technical tools needed in the proof. In Section 4. we will prove the main results. We will end this note by showing some further development after Angehrn and Siu.

The major references of this note are [Dem12], [Laz04].

2   Main results

Theorem 2.1. Let X be a smooth projective variety of dimension n, and let L be an ample divisor on X. Fix a point $x \in X$, and assume that

$$\left(L^{\dim Z} \cdot Z\right) > \binom{n+1}{2}^{\dim Z} = M^{\dim Z}$$

for every irreducible subvariety $Z \subseteq X$ passing through x (including of course X itself). Then $K_X + L$ is free at x, i.e. $\mathcal{O}_X(K_X + L)$ has a section which does not vanish at x.

In particular, if $L \equiv_{\mathrm{num}} mA$ for some $m > \binom{n+1}{2}$ and some ample A, then $K_X + L$ is free.

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3 Technical tools

3.1 Producing multiplier ideal vanishing at given point

The theorem below shows that the higher the multiplicity of a divisor, the deeper the multiplier ideal sheaf associated to the divisor.

Lemma 3.1. Let X be smooth variety with $D \geq 0$ and effective $\mathbb{R}$-divisor, then

1. If $\mathrm{mult}_x D < 1$ then $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$,

2. If $\mathrm{mult}_x D \geq n$, then $\mathcal{J}(D) \subset \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the corresponding maximal ideal.

Proof. □

3.2 Using divisor with large volume to produce a section with high multiplicity

Lemma 3.2 (Divisor with high multiplicity in the linear system with large volume)**.** Let $x \in V$ be a smooth point on an irreducible projective variety of dimension d, $0 < a \in \mathbb{Q}$ and A an ample Cartier divisor on V such that the volume is lowerbound $A^d = \mathrm{Vol}(A) > a^d$.

Then, for any $k \gg 0$, there exists a divisor $A_k \in |kA|$ such that $\mathrm{mult}_x(A) > ka$.

Proof. □

3.3 Cutting down the LC locus using some divisor with high multiplicity

Lemma 3.3 (Cutting down the LC locus using divisor with high multiplicity)**.** Let X be a smooth variety and $x \in X$ a fixed point. Consider an effective **Q**-divisor D on X with $c(D; x) = 1$, and suppose that $Z = \mathrm{LC}(D; x)$ is irreducible of dimension d at x.

Fix a general smooth point $y \in Z$, and let B be any effective $\mathbb{Q}$-divisor on X, with $Z \nsubseteq \mathrm{Supp}\, B$. Assume that
$$\mathrm{mult}_y (Z; B_Z) > d = \dim Z,$$
where B_Z denotes the restriction of B to Z, and as indicated the multiplicity is computed on Z.

Then

1. For $0 < \varepsilon \ll 1$,
$$\mathcal{J}(X, (1 - \varepsilon)D + B)$$
   is non-trivial at y.

2. If moreover $\mathcal{J}(X, D + B) = \mathcal{J}(X, D)$ away from Z, then
$$Z' = \mathrm{Zeroes}(\mathcal{J}((1 - \varepsilon)D + B))$$
   is a proper subvariety of Z in a neighborhood of y.

Proof. □

3.4 Peturbation of LC locus

Lemma 3.4. Let X be a nonsingular variety, $x \in X$ a fixed point and D an effective Q-divisor whose support contains x. Then there exists an effective **Q**-divisor E such that

$$\mathrm{LC}((D + tE); x) \subseteq X$$

is irreducible at x for all $0 < t \ll 1$.

Moreover if X is projective and L is any ample divisor, then one can take $E \equiv_{\mathrm{num}} pL$ for some $p > 0$.

Proof. □

4 Proof of the theorem

4.1 Construct divisor D_1

Proof. □

4.2 Construct divisor D_i inductively

Let $Z = V(\mathcal{J}(D_1))$ and $d = \dim Z$. We need to inductively find a sequence of divisors $\{D_i\}$, such that

$$Z = Z_1 = V(\mathcal{J}(D_1)) \supsetneq V(\mathcal{J}(D_2)) \supsetneq \ldots,$$

We hope to repeat the construction process like D_1. The provblem is $x \in Z_1$ needs not to be smooth point of Z_1.

The idea to solve this problem is to use Lemma 3.3. And we need to find some divisor B_0 s.t. (a) $Z \nsubseteq \mathrm{Supp}\, B$, (b) $\mathrm{mult}_y (Z; B_Z) > d = \dim Z$, where B_Z denotes the restriction of B to Z, and as indicated the multiplicity is computed on Z, (c) $\mathcal{J}(X, D + B) = \mathcal{J}(X, D)$ away from Z.

Proof. We divide the proof into 3 steps:

Step 1. (Construction of the limiting divisor $\bar{A}_0$ on Z)

Step 2. (Lifting the divisors $\bar{A}_t$ onto X)

Step 3. (Apply the cutting down of LC locus Lemma 3.3 to B_0)

□

4.3 Finish the prood of the main theorem

Proof. Now we find a $\mathbb{Q}$-effective divisor D such that

$$D \equiv \lambda L, \quad \lambda < 1$$

and the log canonical threshold $\mathrm{LCT}(D, x) = 1$ with $x \in X$ being the isolated point of the log canonical locus.

Consider the following short exact sequence induced by the multiplier ideal sheaf $\mathcal{J}(D)$

$$0 \to \mathcal{J}(D) \to \mathcal{O}_X(K_X + L) \to \mathcal{O}_Z(K_X + L) \to 0,$$

where $Z = V(\mathcal{J})$. It will induce

$$0 \to \ldots \to H^0(X, \mathcal{O}_X(K_X + L)) \to H^0(Z, \mathcal{O}_Z(K_X + L)) \to H^1(X, \mathcal{J}(D) \otimes \mathcal{O}_X(K_X + L)) \to \ldots.$$

Since L is sufficient positive and $\lambda < 1$, by Nadel vanishing theorem $H^1(X, \mathcal{J}(D) \otimes \mathcal{O}_X(K_X + L)) = 0$. Thus the section on Z can be lifted to X i.e.

$$H^0(X, \mathcal{O}_X(K_X + L)) \twoheadrightarrow H^0(Z, \mathcal{O}_Z(K_X + L)).$$

Recall that the log canonical locus

$$\mathrm{LC}(D, x) = V(\mathcal{J}(\mathrm{LCT}(D, x) \cdot D))$$

since $\mathrm{LCT}(D, x) = 1$ this means Z is precisely the log canonical locus of D. By assumption x is the isolated point of the LC locus, the section $s \in H^0(Z, \mathcal{O}_Z(K_X + D))$ can take any value at $x \in X$, in particular s can choose to be a section that does not vanish at x. Lift it to $\widetilde{s} \in H^0(X, \mathcal{O}_X(K_X + D))$ gives the section we want.

$\square$

5 Further remarks

The aim of this section is to introduce some further study of theorem Angehrn and Siu.

5.1 Remarks about the Fujita conjecture

5.2 Application in the birational boundedness problem

5.3 Relation with effective base point free theorem

References

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