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Note I.1 — Iitaka Fibration

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These notes give a first introduction to the Iitaka fibration. The guiding point is that a variety with non-negative Kodaira dimension carries a canonical rational map whose base has dimension $\kappa(X)$ and whose very general fibers have Kodaira dimension zero. As we will see Iitaka fibration is one of the building block of minimal model program. Iitaka fibration is also very useful when dealing with problem with intermediate Kodaira dimension. Iitaka fibration is also very useful in the "divide and rule" construction: it isolates the positive Kodaira dimension on the base and leaves a Kodaira-dimension-zero problem on the general fiber.

Contents

1 Introduction

Let X be a normal projective variety, and suppose for the moment that $\kappa(X) \geq 0$. The pluricanonical systems $|mK_X|$ define rational maps

$$\Phi_{|mK_X|} \colon X \dashrightarrow \mathbb{P}H^0(X, mK_X)^\vee.$$

For m sufficiently large and divisible, these maps stabilize up to birational equivalence. The resulting birational fiber space is the *Iitaka fibration* of X:

$$X' \longrightarrow Y, \qquad \dim Y = \kappa(X), \qquad \kappa(F) = 0$$

for a very general fiber F.

One can package the same idea by the canonical ring

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, mK_X).$$

When this ring is finitely generated, as in the projective klt setting by BCHM [BCHM10], the canonical model is $\operatorname{Proj} R(X, K_X)$ and the Iitaka fibration is the rational map from X to this model. The intrinsic content is not the particular multiple mK_X, but the stable birational fiber space seen by all large pluricanonical systems.

The minimal model program suggests the following coarse dichotomy. If $\kappa(X) < 0$, the expected output is a Mori fiber space. If $\kappa(X) \geq 0$, the expected output is a minimal model whose canonical divisor is semiample, and hence induces the Iitaka fibration:

$$\begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ X & \dashrightarrow & X_{\min} \\ & & \downarrow \\ & & Y. \end{array}$$

In the semiample case, $K_{X_{\min}} \sim_{\mathbb{Q}} g^*A$ for an ample $\mathbb{Q}$-divisor A on Y, and the general fiber has torsion canonical class after passing to the minimal model.

2 Kodaira Maps and Kodaira Dimension

2.1 Kodaira maps

Definition 2.1 (Kodaira map)**.** Let X be a normal projective variety and let D be a Cartier divisor on X with $H^0(X, \mathcal{O}_X(D)) \neq 0$. The complete linear system $|D|$ defines a rational map

$$\Phi_{|D|} \colon X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(D))^\vee.$$

If $s_0, \ldots, s_N$ is a basis of $H^0(X, \mathcal{O}_X(D))$, then away from the base locus this map is

$$x \longmapsto [s_0(x) : \cdots : s_N(x)].$$

Equivalently, x is sent to the hyperplane of sections vanishing at x.

Remark 2.2. The map is undefined exactly at the base locus $\operatorname{Bs}|D|$, where every section of $\mathcal{O}_X(D)$ vanishes. After replacing X by a resolution of the graph of $\Phi_{|D|}$, the Kodaira map becomes a morphism. This replacement is harmless for birational questions such as Kodaira dimension and the Iitaka fibration.

Definition 2.3 (Iitaka dimension of a divisor)**.** Let L be a line bundle on a normal projective variety X. Set

$$\mathbb{N}(X, L) = \{m \in \mathbb{N} \mid H^0(X, L^{\otimes m}) \neq 0\}.$$

If $\mathbb{N}(X, L) = \varnothing$, define $\kappa(X, L) = -\infty$. Otherwise,

$$\kappa(X, L) = \max_{m \in \mathbb{N}(X,L)} \dim \Phi_{|mL|}(X).$$

For a smooth projective variety X, the Kodaira dimension is $\kappa(X) = \kappa(X, K_X)$. For a singular variety, $\kappa(X)$ is defined on a smooth birational model.

2.2 The big case

Theorem 2.4 (Kodaira map of a big divisor)**.** Let X be a proper normal variety and let D be a Cartier divisor. Assume that the section ring

$$R(X,D) = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mD))$$

is generated in degree one and that D is big. Let $\phi = \Phi_{|D|}$ and let $Z = \overline{\phi(X)}$. Then ϕ is birational onto the normal variety Z. Moreover:

1. $Z \setminus \phi(X \setminus \operatorname{Bs}|D|)$ has codimension at least two;
2. every divisorial component of $\operatorname{Bs}|D|$ is contracted by ϕ;
3. if D is nef, then $|D|$ is base point free.

Proof. Replace X by the normalization of the closure of the graph of ϕ. We may assume ϕ is a morphism and

$$|D| = \phi^*|H| + F,$$

where H is the hyperplane class on Z and F is the fixed part.

Because the section ring is generated in degree one, the sections of mD are exactly the sections coming from mH on Z. Since D is big, $h^0(X, mD)$ grows like $m^{\dim X}$; hence $h^0(Z, mH)$ also grows like $m^{\dim X}$. Therefore $\dim Z = \dim X$, so ϕ is generically finite.

Let $Z' \to Z$ be the normalization of Z in the function field of X. For $m \gg 0$,

$$H^0(Z, \mathcal{O}_Z(mH)) \subseteq H^0(Z', \mathcal{O}_{Z'}(mp^*H)) \subseteq H^0(X, \mathcal{O}_X(mD)) = H^0(Z, \mathcal{O}_Z(mH)).$$

Thus the inclusions are equalities. Since high multiples recover the projective model, $Z = Z'$ is normal. The Stein factorization then shows that ϕ is birational.

Now let B be a divisor on X whose image in Z is also divisorial. By twisting with a sufficiently positive divisor on Z, one finds sections of $m\phi^*H + B$ not vanishing along B. Hence such a B cannot be contained in the fixed divisor of $|mD|$. It follows that divisorial components of the base locus are ϕ-exceptional, and their images have codimension at least two.

Finally, if D is nef, then the fixed divisor F is both effective and ϕ-exceptional, while $D - \phi^*H = F$ is ϕ-nef. By the negativity lemma, $F = 0$. Thus $|D| = \phi^*|H|$ is base point free. □

3 The Iitaka Fibration

3.1 Existence

Theorem 3.1 (Iitaka fibration theorem, cf. Ueno [Uen75])**.** Let X be a normal projective variety and let L be a line bundle with $\kappa(X, L) > 0$. Then for all sufficiently large $m \in \mathbb{N}(X, L)$, the Kodaira maps

$$\Phi_{|mL|} : X \dashrightarrow Y_m$$

are birationally equivalent to a fixed algebraic fiber space

$$f_\infty \colon X_\infty \longrightarrow Y_\infty$$

of normal varieties. Moreover

$$\dim Y_\infty = \kappa(X, L),$$

and if F is a very general fiber of f_∞, then

$$\kappa(F, L_\infty|_F) = 0, \qquad L_\infty = u_\infty^* L.$$

Definition 3.2. The birational equivalence class of $f_\infty \colon X_\infty \to Y_\infty$ is called the *Iitaka fibration* of L. If $L = K_X$ on a smooth model of X, it is called the Iitaka fibration of X.

Proof. Replacing L by a positive power does not change the desired birational fibration, so we may assume that $L = \mathcal{O}_X(D)$ for an effective Cartier divisor D. Choose m_0 such that for every $m \geq m_0$ with $m \in \mathbb{N}(X, L)$, the image

$$Y_m := \Phi_{|mL|}(X)$$

has dimension $\kappa(X, L)$ and $\mathbb{C}(Y_m)$ is algebraically closed in $\mathbb{C}(X)$. Fix such an m.

Let X_∞ be a nonsingular model of the graph of $\Phi_{|mL|}$, with birational morphism $u_\infty \colon X_\infty \to X$. Let

$$f_\infty \colon X_\infty \to Y_\infty$$

be the induced morphism to the normalization of Y_m. Since pullback by u_∞ identifies

$$H^0(X, \ell L) \simeq H^0(X_\infty, \ell L_\infty), \qquad L_\infty = u_\infty^* L,$$

we may prove the fiber statement on this model.

Set $\mathcal{L} = mL_\infty$. Let $N + 1 = h^0(X_\infty, \mathcal{L})$, so $Y_\infty \subset \mathbb{P}^N$. Choose a hyperplane section H_λ of Y_∞, and put $U_\lambda = Y_\infty \setminus H_\lambda$. For each $n > 0$, the sheaf

$$f_{\infty *}\mathcal{L}^{\otimes n}$$

is coherent. Since U_λ is affine, its restriction to U_λ is generated by finitely many sections

$$\psi_0, \ldots, \psi_M \in H^0(f_\infty^{-1}(U_\lambda), \mathcal{L}^{\otimes n}).$$

After allowing poles along H_λ, there is an integer $e > 0$ such that these sections may be regarded as global sections of

$$\mathcal{L}^{\otimes n} \otimes f_\infty^* \mathcal{O}_{Y_\infty}(enH_\lambda).$$

Multiplying by a section cutting out enH_λ turns them into sections of $\mathcal{L}^{\otimes (e+1)n}$. Together with the degree n monomials in a basis of $H^0(X_\infty, \mathcal{L})$, they define a rational map

$$h^{(n)} \colon X_\infty \dashrightarrow V_n.$$

By construction,

$$\mathbb{C}(Y_\infty) \subseteq \mathbb{C}(V_n) \subseteq \mathbb{C}(Y_{(e+1)nm}).$$

The right-hand side equals $\mathbb{C}(Y_\infty)$ because Y_∞ was chosen at the maximal Kodaira dimension with algebraically closed function field. Hence V_n is birational to Y_∞.

We now restrict to fibers. Remove from Y_∞ the union of the hyperplane H_λ, the singular locus, the bad locus for generic smoothness and base change for the countably many sheaves $f_{\infty *}\mathcal{L}^{\otimes n}$, and the exceptional loci where the birational maps $V_n \dashrightarrow Y_\infty$ are not isomorphisms. The complement is dense. If y lies in this complement and $F = f_\infty^{-1}(y)$, then the restriction of $h^{(n)}$ to F is the Kodaira map associated to $\mathcal{L}^{\otimes n}|_F$, but its image is the fiber of a birational map $V_n \dashrightarrow Y_\infty$ over y, hence a point. Therefore

$$h^0(F, \mathcal{L}^{\otimes n}|_F) = 1$$

for every $n > 0$. Since D is effective, the same conclusion forces

$$\kappa(F, L_\infty|_F) = 0.$$

The construction is birationally equivalent to the Kodaira maps $\Phi_{|kL|}$ for all sufficiently large $k \in \mathbb{N}(X, L)$, because the same argument applied to such k gives the same algebraically closed subfield of $\mathbb{C}(X)$. This proves the theorem. □

Remark 3.3. If L is big, then $\kappa(X, L) = \dim X$, and the Iitaka fibration is birational. Thus Theorem 3.1 interpolates between the big case and the Kodaira-dimension-zero case.

3.2 Comments on the Statement

There are two small technical points in Theorem 3.1 that are worth keeping visible.

First, the phrase *very general fiber* means that the assertion holds away from a countable union of proper subvarieties of the base. This is stronger than saying that it holds over a single Zariski open set, and it is natural in the proof: for each positive integer n, one removes the bad locus where base change, smoothness, or the comparison with the n-th Kodaira map fails. Taking the union over all n produces a countable exceptional set. The reward is that on the remaining fibers one controls all plurisection spaces simultaneously.

Second, the Iitaka fibration is a birational object. It is not a single preferred morphism from the original variety X unless extra finite-generation or semiampleness hypotheses are imposed. What is canonical is the resulting subfield

$$\mathbb{C}(Y_\infty) \subseteq \mathbb{C}(X)$$

generated by the large plurisection maps. Replacing X or Y_∞ by birational models changes the presentation but not this field extension. This is why the theorem is naturally stated up to birational equivalence.

Finally, the equality $\dim Y_\infty = \kappa(X, L)$ is the reason the general fiber has L-Kodaira dimension zero. If a very general fiber still carried positive L-Kodaira growth, then the large Kodaira maps would see additional directions along the fiber, and the image dimension of the total space would exceed $\kappa(X, L)$. The Iitaka fibration is precisely the map obtained after all such positive directions have been moved into the base.

3.3 Universal property

Proposition 3.4 (Universal property)**.** Let $\lambda\colon X \dashrightarrow W$ be a rational fiber space of normal projective varieties. Suppose that for a very general fiber G of λ,

$$\kappa(G, L|_G) = 0.$$

Then λ factors birationally through the Iitaka fibration of L.

Proof. Let $f_\infty\colon X_\infty \to Y_\infty$ be the Iitaka fibration after resolving indeterminacies, and resolve λ on the same model. Since $\kappa(G, L|_G) = 0$, the restrictions of all sufficiently divisible Kodaira maps of L to G have zero-dimensional image. Therefore the large Kodaira maps are constant on very general fibers of λ.

But the Iitaka fibration is birationally represented by these large Kodaira maps. Hence the rational map to Y_∞ is constant along the fibers of λ, which is precisely the assertion that it factors through W up to birational equivalence. ☐

Remark 3.5. This explains why the Iitaka fibration is the maximal fibration whose very general fiber has L-Kodaira dimension zero. Any other such fibration must be obtained from it by collapsing some birational information on the base.

3.4 The canonical case

When $L = K_X$, Theorem 3.1 gives a birational fiber space

$$f\colon X' \to Y, \qquad \dim Y = \kappa(X), \qquad \kappa(F) = 0$$

for a very general fiber F. Thus the Iitaka fibration decomposes the study of X into:

1. the base Y, which carries the canonical growth of X;
2. the very general fiber F, which has Kodaira dimension zero;
3. the variation of the fibers over the base.

This is the basic "divide and rule" perspective: many questions about a variety of intermediate Kodaira dimension can be attacked by understanding Kodaira-dimension-zero fibers and the geometry of the base.

3.5 Relative Iitaka Fibrations

Theorem 3.6 (Relative Iitaka fibration, cf. Nakayama [Nak04])**.** Let $f\colon X \to Z$ be a projective morphism of normal varieties, and let D be a $\mathbb{Q}$-Cartier divisor on X. Let F be a general fiber of f and assume

$$r := \kappa(D|_F) \geq 0.$$

Then, after replacing X by a resolution $h\colon W \to X$, there is a contraction $g\colon W \to T$ over Z such that, if V is a general fiber of $T \to Z$ and G is a general fiber of g, then

$$\dim V = r, \qquad \kappa(h^* D|_G) = 0.$$

$$\begin{array}{ccc} W & \xrightarrow{\ h\ } & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Z. \end{array}$$

Proof. Choose a sufficiently ample divisor A on Z so that

$$H^0(X, \mathcal{O}_X(mD + f^*A)) \neq 0$$

for some divisible $m > 0$, and set $L = D + f^*A$. Fujita's Iitaka dimension identity gives

$$\kappa(L) = \kappa(D|_F) + \dim Z = r + \dim Z.$$

Take the ordinary Iitaka fibration of L on a resolution of X:

$$\varphi\colon W \longrightarrow S.$$

We show that this fibration factors over Z. Let $\psi = f \circ h\colon W \to Z$. For a sufficiently divisible multiple, the linear system defining φ has the form

$$|kmL| = |M| + F_{\text{fix}}, \qquad M \sim \varphi^*H$$

for an ample divisor H on S. Since $mD + f^*A$ has a nonzero section, after pullback to W we may write

$$mh^*D + \psi^*A \sim B \geq 0.$$

Thus

$$mh^*L \sim B + (m-1)\psi^*A.$$

For a large multiple k, subtracting the fixed part of $|kmh^*L|$ gives

$$M \sim N + k(m-1)\psi^*A$$

with $N \geq 0$. If C is a curve contracted by φ and chosen away from $\operatorname{Supp} N$, then

$$0 = M \cdot C = N \cdot C + k(m-1)\psi^*A \cdot C.$$

Both terms on the right are non-negative, so $\psi^*A \cdot C = 0$. Since A is ample, $\psi(C)$ is a point. Thus curves contracted by φ are contracted by ψ on a dense open set, and the rigidity lemma gives a morphism $S \to Z$. We write this relative model as T, and write $g\colon W \to T$ for the induced contraction.

Finally,

$$\dim T = \kappa(L) = r + \dim Z,$$

so a general fiber V of $T \to Z$ has dimension r. On a general fiber G of g, the ordinary Iitaka theorem gives $\kappa(h^*L|_G) = 0$. Since G lies inside a fiber of ψ, the divisor ψ^*A restricts trivially to G. Hence $h^*L|_G = h^*D|_G$, and $\kappa(h^*D|_G) = 0$. □

4 Applications

4.1 Good Minimal Models and Abundance

The first application explains why the Iitaka fibration is not just a map attached to a linear system, but also a mechanism for reducing the abundance problem. Once the base has dimension κ and the very general fiber has Kodaira dimension zero, the remaining issue is whether the numerical positivity on the fiber disappears. If it does, one can often appeal to the known numerical-dimension-zero case and then lift the result back to the total space.

Theorem 4.1 (Good minimal model versus abundance)**.** For pseudo-effective klt pairs, the existence of good minimal models is equivalent to abundance:

$$\kappa_\sigma(K_X + \Delta) = \kappa(K_X + \Delta).$$

Proof. Assume first that (X, Δ) has a good minimal model (X', Δ'). Then $K_{X'} + \Delta'$ is semiample, so it defines a morphism

$$f\colon X' \to Z = \operatorname{Proj} R(X', K_{X'} + \Delta')$$

and

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} f^*A$$

for an ample $\mathbb{Q}$-divisor A on Z. Since Kodaira dimension and numerical Kodaira dimension are birational invariants for this purpose, and since semiample divisors are abundant,

$$\kappa_\sigma(K_X + \Delta) = \kappa_\sigma(K_{X'} + \Delta') = \dim Z = \kappa(K_{X'} + \Delta') = \kappa(K_X + \Delta).$$

Conversely assume abundance. If $\kappa_\sigma(K_X + \Delta) = \dim X$, then $K_X + \Delta$ is big and BCHM gives a good minimal model. If $\kappa_\sigma(K_X + \Delta) = 0$, the numerical-dimension-zero case gives a good minimal model. Thus we may assume

$$0 < \kappa(K_X + \Delta) = \kappa_\sigma(K_X + \Delta) < \dim X.$$

Let

$$f\colon X \dashrightarrow Z$$

be a birational model of the Iitaka fibration of $K_X + \Delta$, and let F be a very general fiber. The Iitaka fibration gives

$$\dim Z = \kappa(K_X + \Delta).$$

By the easy addition formula for numerical Kodaira dimension and its converse in this setting,

$$\kappa_\sigma(K_X + \Delta) = \kappa_\sigma(K_F + \Delta|_F) + \dim Z.$$

Therefore

$$\kappa_\sigma(K_F + \Delta|_F) = 0.$$

The numerical-dimension-zero case gives a good minimal model for the very general fiber. Applying the theorem that a variety whose Iitaka-fibration general fiber has a good minimal model itself has a good minimal model, one obtains a good minimal model for (X, Δ). □

4.2 Asymptotic Definition of Kodaira Dimension

Historically, Iitaka dimension was formulated through the growth of the number of sections. The Iitaka fibration gives the geometric reason for this growth: sections grow like sections on the base, while the very general fibers contribute only bounded-dimensional spaces of sections.

Theorem 4.2 (Ueno's asymptotic estimate)**.** Let V be a normal projective variety and let D be a Cartier divisor with $\kappa(D, V) \geq 0$. Let d be the greatest common divisor of $\mathbb{N}(D, V)$. Then there are positive constants α, β and an integer m_0 such that

$$\alpha m^{\kappa(D,V)} \leq h^0(V, \mathcal{O}_V(mdD)) \leq \beta m^{\kappa(D,V)}$$

for all $m \geq m_0$.

Proof. It is enough to prove the result after replacing D by a sufficiently large effective multiple $E \sim ndD$. Indeed, writing $m = m_1 n + m_2$ with $0 \leq m_2 < n$, the estimates for E imply the corresponding estimates for D after changing the constants. Thus assume D is effective and that $\mathbb{N}(D,V) = \mathbb{N}$.

Choose D so that the Kodaira map

$$f = \Phi_{|D|} \colon V \to W$$

is a morphism, $\dim W = \kappa(D,V)$, and $\mathbb{C}(W)$ is algebraically closed in $\mathbb{C}(V)$. Let H_λ be a hyperplane section of $W \subset \mathbb{P}^N$, and let $E_\lambda \in |D|$ be the corresponding divisor on V. Write

$$E_\lambda = E_\lambda^* + F, \qquad E_\lambda^* = f^* H_\lambda,$$

where F is the fixed part of $|D|$.

For the lower bound,

$$h^0(V, mD) = h^0(V, mE_\lambda) \geq h^0(V, mE_\lambda^*) \geq h^0(W, mH_\lambda).$$

Since $\dim W = \kappa(D,V)$ and H_λ is ample on W, the last term is bounded below by $\alpha m^{\kappa(D,V)}$ for $m \gg 0$.

For the upper bound, decompose the fixed part as

$$F = L + F^*,$$

where L is the sum of the components dominating W and F^* is vertical over W. Let

$$B_m = L_m + F_m^* \in |mD|$$

be a general member, with L_m the sum of components dominating W and F_m^* vertical. By the fiber statement in the Iitaka fibration theorem, for very general $w \in W$,

$$h^0(V_w, \mathcal{O}_{V_w}(mD|_{V_w})) = 1.$$

Restricting the linearly equivalent divisors B_m and mE_λ to such fibers gives

$$L_m|_{V_w} = mL|_{V_w}.$$

Since all components of L_m and L dominate W, this equality on very general fibers implies $L_m = mL$. Thus the horizontal part mL is fixed in $|mD|$, and

$$h^0(V, mD) = h^0(V, mF^* + mE_\lambda^*).$$

Every component of F^* maps to a proper subvariety of W, so after replacing an effective Cartier divisor H on W by a large multiple we have

$$F^* \leq f^* H.$$

Therefore

$$h^0(V, mD) \leq h^0(V, mf^*(H + H_\lambda)) = h^0(W, m(H + H_\lambda)).$$

The divisor $H + H_\lambda$ is effective on the $\kappa(D,V)$-dimensional variety W, so the elementary growth estimate for effective divisors gives

$$h^0(W, m(H + H_\lambda)) \leq \beta m^{\kappa(D,V)}$$

for $m \gg 0$. This proves the upper bound. □

Corollary 4.3. For a line bundle L on a normal projective variety,

$$\kappa(X, L) = \limsup_{m\to\infty} \frac{\log h^0(X, L^{\otimes m})}{\log m}.$$

4.3 Easy Addition Formula

Theorem 4.4 (Easy addition)**.** Let $f\colon V \to W$ be a fiber space between nonsingular projective varieties, and let D be a Cartier divisor on V. Then there is a dense open subset $U \subseteq W$ such that for every $w \in U$,

$$\kappa(D, V) \le \dim W + \kappa(D_w, V_w), \qquad V_w = f^{-1}(w), \quad D_w = D|_{V_w}.$$

Proof. If $\kappa(D, V) = -\infty$, there is nothing to prove. Fix $m \in \mathbb{N}(D, V)$ and set

$$\mathcal{L}_m = f_*\mathcal{O}_V(mD).$$

By generic freeness and base change, there is a dense open subset $U^{(m)} \subseteq W$ such that $\mathcal{L}_m|_{U^{(m)}}$ is locally free, V_w is nonsingular, and

$$\mathcal{L}_m \otimes \mathbb{C}(w) \simeq H^0(V_w, \mathcal{O}_{V_w}(mD_w))$$

for every $w \in U^{(m)}$.

Let

$$g^{(m)}\colon \mathbb{P}(\mathcal{L}_m|_{U^{(m)}}) \to U^{(m)}$$

be the associated projective bundle. The natural evaluation map defines a relative Kodaira map

$$h^{(m)}\colon f^{-1}(U^{(m)}) \dashrightarrow \mathbb{P}(\mathcal{L}_m|_{U^{(m)}})$$

over $U^{(m)}$. By the base change isomorphism above, the restriction

$$h^{(m)}_w\colon V_w \dashrightarrow \mathbb{P}(\mathcal{L}_m \otimes \mathbb{C}(w))$$

is exactly the Kodaira map $\Phi_{|mD_w|}$ for $w \in U^{(m)}$.

Shrinking $U^{(m)}$, we may assume that the dimension of the image of this fiberwise Kodaira map is constant. Then

$$\dim h^{(m)}(V) = \dim W + \dim \Phi_{|mD_w|}(V_w)$$

for w in a dense open subset. This is simply the fiber-dimension formula applied to the image of the relative Kodaira map over the base.

On the other hand, the canonical isomorphism

$$H^0(W, \mathcal{L}_m) = H^0(V, \mathcal{O}_V(mD))$$

gives a rational map

$$\mathbb{P}(\mathcal{L}_m) \dashrightarrow \mathbb{P}H^0(V, \mathcal{O}_V(mD))^\vee$$

whose composition with $h^{(m)}$ is the absolute Kodaira map $\Phi_{|mD|}$. Hence the image of $\Phi_{|mD|}$ is dominated by $h^{(m)}(V)$, so

$$\dim \Phi_{|mD|}(V) \le \dim h^{(m)}(V) = \dim W + \dim \Phi_{|mD_w|}(V_w).$$

Now choose m sufficiently large so that $\dim \Phi_{|mD|}(V) = \kappa(D, V)$. Since

$$\dim \Phi_{|mD_w|}(V_w) \le \kappa(D_w, V_w),$$

we obtain

$$\kappa(D, V) \le \dim W + \kappa(D_w, V_w)$$

for w in a dense open subset of W. □

4.4 Finite Generation of the Canonical Ring

The previous applications use the Iitaka fibration as a birational map. Finite generation is stronger: it says that the stable pluricanonical maps are controlled by one finitely generated graded algebra. The proof below is written in the form suggested by Fujino's analytic proof [Fuj22]: the Iitaka fibration is the device that reduces finite generation to the big case on the base.

Theorem 4.5 (Finite generation of the canonical ring)**.** Let (X, Δ) be a projective klt pair with $K_X + \Delta$ $\mathbb{Q}$-Cartier. Then the log canonical ring

$$R(X, K_X + \Delta) = \bigoplus_{m \ge 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is finitely generated.

Proof. We explain the reduction in the pseudo-effective case; the big case is the finite generation theorem of BCHM [BCHM10]. After taking a log resolution and replacing by a suitable birational model, assume that X is smooth and $\operatorname{Supp} \Delta$ has simple normal crossings.

Let

$$f \colon X \dashrightarrow Z$$

be the Iitaka fibration of $K_X + \Delta$. After resolving the indeterminacy, we may assume that

$$f \colon X \to Z$$

is a morphism with connected fibers and that Z is smooth. The defining property of the Iitaka fibration is

$$\dim Z = \kappa(K_X + \Delta), \qquad \kappa((K_X + \Delta)|_F) = 0$$

for a very general fiber F.

The canonical bundle formula expresses the adjoint divisor upstairs in terms of data on the base:

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

after replacing Z by a suitable birational model. Here B_Z is the discriminant part and M_Z is the moduli part. Consequently, after taking a sufficiently divisible Veronese subring, the canonical ring of (X, Δ) is identified with

$$\bigoplus_{m \geq 0} H^0(Z, \mathcal{O}_Z(mk(K_Z + B_Z + M_Z)))$$

for some $k > 0$.

This is the point where the Iitaka fibration is essential. Since Z is the base of the Iitaka fibration,

$$\kappa(K_Z + B_Z + M_Z) = \dim Z.$$

Thus $K_Z + B_Z + M_Z$ is big. In other words, the Iitaka fibration has converted the original non-big divisor $K_X + \Delta$ on X into a big divisor on its canonical base.

It remains to replace the generalized pair on Z by an ordinary klt pair. Since M_Z is nef and $K_Z + B_Z + M_Z$ is big, write

$$K_Z + B_Z + M_Z \sim_{\mathbb{Q}} A + G$$

with A ample and $G \geq 0$. For $0 < \varepsilon \ll 1$, the pair

$$(Z, \Delta_Z), \qquad \Delta_Z = B_Z + \varepsilon G + H$$

is klt for a general small effective $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} M_Z + \varepsilon A$, and

$$K_Z + \Delta_Z \sim_{\mathbb{Q}} (1 + \varepsilon)(K_Z + B_Z + M_Z).$$

By the truncation principle for finite generation, it is enough to prove finite generation for

$$R(Z, K_Z + \Delta_Z).$$

But $K_Z + \Delta_Z$ is big and (Z, Δ_Z) is klt, so BCHM applies. Hence $R(Z, K_Z + \Delta_Z)$ is finitely generated, and therefore so is $R(X, K_X + \Delta)$. □

Corollary 4.6 (Iitaka model as Proj)**.** When the canonical ring is finitely generated,

$$X \dashrightarrow \operatorname{Proj} R(X, K_X + \Delta)$$

is the Iitaka fibration of $K_X + \Delta$ up to birational equivalence. After passing to a Veronese subring generated in degree one, a single pluricanonical system realizes the stabilized Iitaka map.

Remark 4.7. Thus the Iitaka fibration appears twice in finite generation. First, it is the geometric output encoded by $\operatorname{Proj} R(X, K_X + \Delta)$. Second, in the proof, it is the reduction map that moves the problem from a divisor of intermediate Kodaira dimension on X to a big divisor on the base Z.

5 Further Directions

Remark 5.1 (Abundance)**.** The Iitaka fibration is the natural map expected to be induced by a semiample canonical divisor. Thus abundance can be read as the statement that the canonical divisor descends from the base of the Iitaka fibration; see Nakayama [Nak04] and the finite-generation framework of BCHM [BCHM10].

Remark 5.2 (Good fibers)**.** Lai's theorem [Lai11] shows that, for terminal projective varieties, good minimal models of the general fibers of the Iitaka fibration imply a good minimal model of the total space. This is a standard way to reduce an intermediate Kodaira dimension problem to the Kodaira-dimension-zero fiber.

Remark 5.3 (Nef cotangent and nef anti-canonical geometry)**.** In positivity problems, the Iitaka fibration often becomes more rigid. For example, under nef cotangent and semiampleness hypotheses, one expects the fibration to have torus-like fibers and a base with positive canonical class. This is a useful direction for connecting Iitaka fibrations with structure theorems for varieties with special tangent or anti-canonical positivity.

Remark 5.4 (Iitaka conjecture)**.** Iitaka's conjecture predicts $\kappa(X) \geq \kappa(F) + \kappa(Y)$ for an algebraic fiber space $f\colon X \to Y$. The relative Iitaka fibration is one of the basic tools for separating the Kodaira growth along the fiber from the growth on the base; the maximal Albanese dimension case is treated by Hacon–Popa–Schnell [HPS18].

Remark 5.5 (Effective and bounded Iitaka fibrations)**.** The existence theorem is qualitative: it says that the maps $\Phi_{|mK_X|}$ eventually stabilize. Effective Iitaka fibration and boundedness problems ask for uniform control of such an m, or of the varieties appearing in families with fixed Iitaka data; see Birkar–Zhang [BZ16].

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