I.3 — MRC Fibration

Fibrations in Birational Geometry Reading Notes • Summer 2026

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Big picture

In this note we give a self-contained introduction to the MRC fibration, which separates a variety into a rationally connected piece and a base variety that is not uniruled. What makes the MRC fibration special among the three fibrations is that it has a projective birational model, even when the total space is only compact Kähler. This single fact is what makes the MRC fibration particularly useful in the analytic MMP and in structure theorems for Kähler varieties with nef anti-canonical divisor. We first introduce some elementary properties of rationally connected varieties (which serve as the fibers of the MRC fibration). We then discuss the MRC fibration itself, together with some applications, and end by pointing out some further directions.

Convention 0.1. Without further mention, we always work with complex varieties, or varieties over an algebraically closed field of characteristic 0.

Contents

1 Rationally Connected Varieties

Rationally connected varieties can be viewed as the key building block of the MRC fibration, as the general fibers of the MRC fibration are rationally connected.

Definition 1.1 (Rationally connected variety, cf. [Kol96]). A variety X is *rationally connected* if for every two points $x, y \in X$ there is a morphism $f : \mathbb{P}^1 \to X$ with $x, y \in f(\mathbb{P}^1)$. Equivalently (for X smooth projective over $\mathbb{C}$), for every $m > 0$ and every $x_1, \ldots, x_m \in X$ there is a rational curve through $x_1, \ldots, x_m$.

Definition 1.2 (Separably rationally connected, very free curve). X is *separably rationally connected* if there is a morphism $f : \mathbb{P}^1 \to X$ that is *very free*, i.e.

$$f^*T_X = \bigoplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i), \qquad a_i \geq 1.$$

Like uniruled varieties, rationally connected varieties admit several equivalent characterizations.

Proposition 1.3 ([Kol96]). *Let X be a smooth compact complex variety over $\mathbb{C}$. The following are equivalent.*
(1) X is rationally connected,
(2) There exists an irreducible variety Y and a map $\mathbb{P}^1 \times Y \to X$ that is non-constant on $\mathbb{P}^1$ such that

$$\mathbb{P}^1 \times \mathbb{P}^1 \times Y \dashrightarrow X \times X$$

is dominant,
(3) There exists a very free rational curve $f : \mathbb{P}^1 \to X$,
(4) For a general collection of $r + 1$ points, there exists an r-free rational curve passing through those points.

Due to limited time, we omit the proof. We also mention that rational chain connectedness is strictly weaker in general, but coincides with rational connectedness for smooth projective varieties over a field of characteristic 0 [Kol96, Thm. 3.10]. The proof requires Kollár's smoothing-of-chains lemma and is quite technical; we do not present it here.

We next mention one characteristic property[1] of a rationally connected variety: it carries no non-trivial global holomorphic k-forms.

Theorem 1.4 (No holomorphic forms, cf. [Kol96, Cor. 3.8]). *If X is a smooth separably rationally connected projective variety, then*

$$H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0 \qquad \text{for every } m > 0.$$

In particular X carries no non-zero holomorphic n-forms for any $n \geq 1$.

[1] The converse is still open — the so-called Mumford conjecture.

Proof. Let $\omega \in H^0\left(X, \left(\Omega_X^n\right)^{\otimes m}\right)$. If $f : \mathbb{P}^1 \to X$ is a very free rational curve, then $f^*\Omega_X$ is a direct sum of negative line bundles so the restriction $f^*\left(\Omega_X^n\right)^{\otimes m}$ has no sections for any $m, n \geq 1$. Thus $f^*\omega = 0$ but very free curves cover X so $\omega = 0$. ∎

Rationally connected varieties enjoy an important vanishing property.

Corollary 1.5 (Vanishing result for RC varieties, cf. [Kol96, Lem. 4.4])**.** *If X is a smooth rationally connected variety, then*

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0, \qquad \mathrm{Pic}(X) \cong H^2(X, \mathbb{Z}).$$

Proof. By Theorem 1.4, $H^0(X, \Omega_X^i) = 0$ for $i = 1, 2$ (using $\Omega_X^{\otimes 2} = S^2\Omega_X \oplus \wedge^2\Omega_X$ to extract $H^0(X, \Omega_X^2) = 0$ from the vanishing of $H^0(X, \Omega_X^{\otimes 2})$). By Hodge symmetry, $H^i(X, \mathcal{O}_X) \cong \overline{H^0(X, \Omega_X^i)} = 0$ for $i = 1, 2$. The exponential sequence

$$H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

then forces $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$. ∎

Rational connectedness is a birational invariant.

Proposition 1.6 ([Kol96, Proposition 3.3.3])**.** *Rational connectedness is a birational invariant. More precisely, let $\phi : X \dashrightarrow Y$ be a dominant rational map of varieties with Y proper. If X is rationally connected, then so is Y.*

Proof. Let $U \subset X$ be a dense open subset where ϕ is a morphism. Since ϕ is dominant, a general point of Y lies in the image $\phi(U)$: given two general points $y_1, y_2 \in Y$, there exist $x_1, x_2 \in U$ with $\phi(x_i) = y_i$. Since X is rationally connected, there exists a rational curve $f : \mathbb{P}^1 \to X$ passing through them, with $f(0) = x_1$, $f(\infty) = x_2$. If we compose ϕ with f, we get a meromorphic map

$$\phi \circ f : \mathbb{P}^1 \dashrightarrow Y.$$

We can then extend this meromorphic map to $g : \mathbb{P}^1 \to Y$[a] Hence we find a rational curve passing through two general point y_1, y_2. ∎

[a]This is a standard fact [Uen75]: If X and Y are normal complex varieties and $\varphi : X \dashrightarrow Y$ is a meromorphic mapping, the set $S(\varphi)$ of points of indeterminacy of the meromorphic mapping φ is an analytic subset of X of codimension at least two.

2 Construction of the MRC Fibration

Definition 2.1 (MRC fibration, cf. [Kol96, Proposition 5.3])**.** Let X be a smooth projective manifold/compact Kähler manifold. There exists a smooth projective variety Z, dense open subsets $X_0 \subset X$ and $Z_0 \subset Z$, and a rational map $\mu : X \dashrightarrow Z$ such that the following hold.

1. $\mu|_{X_0} : X_0 \to Z_0$ is a proper morphism.
2. the fibers of μ over Z_0 are rationally connected.
3. For a very general point $z \in Z$, a rational curve in X meets the fiber X_z if and only if it is contained in X_z.

2.1 Existence of the MRC fibrations

This is the central existence result of the theory, due independently to Campana and to Kollár–Miyaoka–Mori.

Theorem 2.2 (Existence of the MRC fibration, cf. [Cam92; KMM92], [Kol96, Thm. 5.4])**.** *Let X be a smooth compact Kähler manifold. There exists a smooth projective variety Z, dense open subsets $X_0 \subset X$ and $Z_0 \subset Z$, and a rational map $\mu : X \to Z$ such that the following hold*

1. *$\mu|_{X_0} : X_0 \to Z_0$ is a proper morphism.*
2. *the fibers of μ over Z_0 are rationally connected.*
3. *For a very general point $z \in Z$, a rational curve in X meets the fiber X_z if and only if it is contained in X_z.*

Moreover, μ is unique up to birational equivalence. Sometimes, we will write the base as $R(X)$ and call it the MRC quotient.

Proof. To prove this, we define a (proalgebraic)[a] equivalence relation RC_X such that $(x_1, x_2) \in \widetilde{\mathrm{RC}}_X$ iff x_1 and x_2 can be connected by a chain of rational curves. We then consider the universal families $U_d \to V_d$, $U_d \to X$ of rational curves of degree $\leq d$ on X. The technical core of the proof is a general machine turning the equivalence relation defined above into a fibration (discussed in the appendix): there is an open subset $X^0 \subset X$ and a proper morphism $\pi : X^0 \to Z^0$ such that if $x_1, x_2 \in X^0$ are very general, then $\pi(x_1) = \pi(x_2)$ iff x_1 and x_2 can be connected by a chain of rational curves. This gives the desired almost holomorphic map, whose fibers are rationally chain connected, hence rationally connected since a general fiber is smooth.

We next show it is maximal. Suppose $\pi' : X \dashrightarrow Z'$ is another rationally chain connected fibration, and let $\Gamma := \mathrm{im}[X \cap X^0 \to Z' \times Z^0]$. We claim that the projection $\Gamma \to Z'$ is birational, so that it is the graph of a map τ. If $x_1, x_2 \in X' \cap X^0$ satisfy $\pi'(x_1) = \pi'(x_2)$, then x_1 and x_2 can be connected by a chain of rational curves (the fiber of π' containing them is rationally chain connected).

If x_1 is very general, this implies $\pi(x_1) = \pi(x_2)$. Hence a very general fiber of π' is contained in a fiber of π, so $\Gamma \to Z'$ is generically one-to-one, hence birational. This gives the rational map $Z' \dashrightarrow Z^0$, i.e. every RCC fibration factors through π. Finally, applying this to two MRCC fibrations in both directions gives uniqueness up to birational equivalence. ∎

[a] One can see this terminology in Kollár's book

Like the Iitaka fibration and the Albanese mapping, the MRC fibration also admits a relative version.

Theorem 2.3 (Relative MRC fibration, cf. [Kol96, Thm. 5.9]). *Let $f: X \to S$ be a smooth proper morphism of complex varieties, S normal. There is an open $X^* \subset X$, an S-scheme Z^*, and a proper morphism $\pi^*: X^* \to Z^*$ such that for every $s \in S$, the induced map $\pi^*_s: X^*_s \to Z^*_s$ is the MRC fibration of X_s. In particular $\dim R(X_s)$ is constant in smooth families.*

Corollary 2.4 (Functoriality, cf. [Kol96, Thm. 5.5]). *Let $f_X: X_1 \dashrightarrow X_2$ be a dominant map of smooth proper varieties in characteristic 0, with MRC fibrations $\pi_i: X_i \dashrightarrow Z_i$. Then there is a rational map $f_Z: Z_1 \dashrightarrow Z_2$ with $\pi_2 \circ f_X = f_Z \circ \pi_1$.*

$$\begin{tikzcd} X_1 \arrow[r, dashed] \arrow[d, dashed] & X_2 \arrow[d, dashed] \\ Z_1 \arrow[r, dashed] & Z_2 \end{tikzcd}$$

Proof. Fix a very general point $z \in Z_1$, and then two very general points $x_1, x_2 \in \pi_1^{-1}(z)$. Choosing z first and x_1, x_2 afterwards, and using that each of the following is a countable intersection of dense open conditions, we may assume simultaneously that:

(a) the fibre $\pi_1^{-1}(z)$ is irreducible and rationally connected;

(b) f_X is defined at x_1 and x_2;

(c) $f_X(x_1)$ and $f_X(x_2)$ are very general points of X_2.

For (c) we use that f_X is dominant: the preimage under f_X of the countable union of proper closed "bad" subsets of X_2 (those where the rational-curve characterization of π_2 fails) is again a countable union of proper closed subsets of X_1, hence avoided by very general points.

By (a) and the characterization of rational connectedness, there is a single *irreducible* rational curve $C \subset \pi_1^{-1}(z)$ containing x_1 and x_2. Since C is irreducible and $x_1 \in C$ lies in the domain of definition of f_X by (b), C meets that domain; hence $f_X|_C$ is defined on a dense open subset of C and $f_X(C) \subset X_2$ is either an irreducible rational curve or a point. In either case $f_X(x_1)$ and $f_X(x_2)$ are joined by a rational curve. By (c) and the characterization of π_2, we conclude

$$\pi_2\big(f_X(x_1)\big) = \pi_2\big(f_X(x_2)\big).$$

We now promote this to a statement about the whole fibre. Set

$$g := \pi_2 \circ f_X \colon X_1 \dashrightarrow Z_2, \qquad w := g(x_1) \in Z_2,$$

a rational map defined on a dense open $U \subset X_1$; write $F := \pi_1^{-1}(z)$. The computation above shows $g \equiv w$ on the very general points of F, which form a dense subset $V \subset F$. Since Z_2 is separated, w is a closed point, so $g^{-1}(w) \cap F$ is closed in $U \cap F$; it contains the dense set V, and $U \cap F$ is a dense open of the *irreducible* variety F. Therefore $g^{-1}(w) \cap F = U \cap F$, i.e.

$$g = w \quad \text{on } U \cap \pi_1^{-1}(z).$$

In other words f_X maps the fibre $\pi_1^{-1}(z)$ into the single fibre $\pi_2^{-1}(w)$ wherever g is defined, and w is independent of the choice of x_1. Setting $f_Z(z) := w$ defines f_Z on very general points of Z_1, with $\pi_2 \circ f_X = f_Z \circ \pi_1$ where all maps are defined.
Let $X_1^0 \subset X_1$ be the (dense open) locus where both π_1 and f_X are defined, and let

$$\Gamma := \overline{\mathrm{im}\Big[X_1^0 \xrightarrow{(\pi_1,\, \pi_2 \circ f_X)} Z_1 \times Z_2\Big]} \subset Z_1 \times Z_2,$$

with $p \colon \Gamma \to Z_1$ the first projection. By Step 1, for very general $z \in Z_1$ every point of $\pi_1^{-1}(z) \cap X_1^0$ has the same image $f_Z(z) \in Z_2$ under g, so the fibre $p^{-1}(z)$ is the single point $(z, f_Z(z))$. Thus p is generically one-to-one, henceâ€"k having characteristic zero, so that the generic fibre is reduced and p is generically separableâ€"birational. Consequently Γ is the graph of a rational map $f_Z \colon Z_1 \dashrightarrow Z_2$ satisfying $\pi_2 \circ f_X = f_Z \circ \pi_1$. Finally f_Z is dominant, since $\pi_2 \circ f_X$ is. ∎

Remark 2.5. This functoriality is special to the MRC (as opposed to MRCC) fibration; it already fails for maximal rationally *chain* connected fibrations of normal varieties.

Corollary 2.6 (MRC fibration detects rational chain connectedness, cf. [Kol96, Cor. 5.15])**.** *X is rationally chain connected if and only if $R(X)$ is a point.*

Proof. If $R(X)$ is a point, the trivial fibration $X \to \{\mathrm{pt}\}$ is already an MRCC fibration whose fibers (namely X itself) are rationally chain connected by definition of the MRC fibration; hence X is rationally chain connected. Conversely, if X is rationally chain connected then $X \to \{\mathrm{pt}\}$ is itself an RCC fibration, and by the maximality property any MRCC fibration $X \dashrightarrow R(X)$ must factor through it; since a point cannot dominate a positive-dimensional variety, $R(X)$ is a point. ∎

Example 2.7 (Fujita, almost holomorphic is sharp)**.** Almost holomorphicity in Theorem 2.2(iii) cannot be upgraded to a genuine morphism in general: there is a projective variety Z of Picard number one, uniruled but not rationally connected (see [**Matsumura**]).

Note that such a Z admits no non-trivial holomorphic fibration. Indeed, suppose there were a fibration

$$f : Z \to B$$

with $\dim Z > \dim B > 0$. Let H_B be an ample divisor on B and $D = f^*H_B$. For a curve C in a fiber (which exists since the general fiber is positive-dimensional),

$$D \cdot C = H_B \cdot f_*C = 0.$$

On the other hand, since $\dim B > 0$, there exists a curve $C' \subset Z$ such that $f(C')$ is a curve on B. By ampleness of H_B we have

$$D \cdot C' = H_B \cdot f_*(C') > 0.$$

However, since Z has Picard number one, $N^1(Z)_{\mathbb{R}} = \mathbb{R} \cdot H$ for an ample class H, so $D \equiv \lambda H$. This is impossible: the two intersection computations above force $\lambda = 0$ and $\lambda > 0$ simultaneously.

3 The theorem of Graber–Harris–Starr

Theorem 3.1 (The MRC quotient is not uniruled, cf. [GHS03, Cor. 1.4])**.** *Let X be any variety and $\phi\colon X \dashrightarrow Z$ its MRC fibration. Then Z is not uniruled.*

Proof. Suppose Z were uniruled: for a general $z \in Z$ there is a non-constant map $g\colon \mathbb{P}^1 \to Z$ through z. Pull back ϕ along g to get $X' = \mathbb{P}^1 \times_Z X \to \mathbb{P}^1$. Its general fiber is rationally connected (being a fiber of ϕ), and the base $\mathbb{P}^1$ is rationally connected, so the total space X' is rationally connected[a].

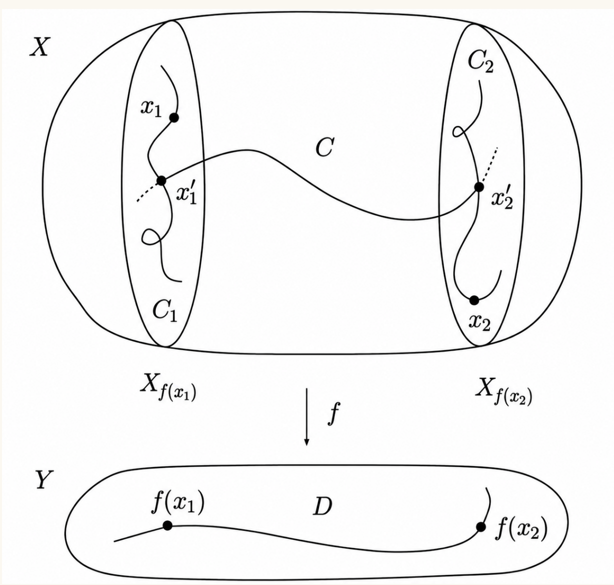

Figure: picture from Kollár's book.

In particular, every point of the fiber X_z lies on a rational curve inside X' that is *not* contained in X_z (a curve dominating $\mathbb{P}^1$ under g), contradicting the maximality property (i) of the MRC fibration. ∎

[a] Although this is geometrically clear from the picture, the proof requires some hard work.

Remark 3.2 (Numerical reformulation)**.** If X is projective, its MRC fibration has projective base Z (Theorem 3.1 is birational-model independent, so we may take Z smooth projective), and the numerical characterization of uniruledness (Boucksom–Demailly–Păun–Peternell) turns Theorem 3.1 into a statement purely about the sign of K_Z:

$$Z \text{ is not uniruled} \iff K_Z \text{ is pseudo-effective.}$$

This is exactly the translation that makes the MRC fibration useful for minimal model program purposes: it produces, canonically, a pseudo-effective-canonical base out of an arbitrary variety.

Theorem 3.3 (MRC fibrational equivalence, cf. [Kol96, pp. 5.6, 5.6.3])**.** *The following two statements about smooth proper varieties in characteristic* 0 *are equivalent:*

(1) the base of every MRC fibration is not uniruled (this is Theorem 3.1, hence unconditionally true given GHS);

(2) if $\rho\colon Y \to \mathbb{P}^1$ is a dominant morphism with rationally connected smooth fibers, then Y is rationally connected.

Proof. (1) ⇒ (2): let $\pi\colon Y \dashrightarrow Z$ be the MRC fibration of Y. Since $\rho\colon Y \to \mathbb{P}^1$ has rationally connected (hence rationally chain connected) fibers, ρ is itself an RCC fibration of Y; by the *maximality* property in the definition of the MRC fibration, π factors through ρ, i.e. there is $\tau\colon \mathbb{P}^1 \dashrightarrow Z$ with $\pi = \tau \circ \rho$. As ρ is dominant, so is τ; but a dominant rational map out of the curve $\mathbb{P}^1$ has image of dimension at most 1, so $\dim Z \le 1$. If $\dim Z = 1$, τ realizes Z as dominated by $\mathbb{P}^1$, hence Z is uniruled, contradicting (1). So Z is a point, and Y, being its own MRC fiber, is rationally connected.
(2) ⇒ (1): let $\pi\colon X \dashrightarrow Z$ be the MRC fibration and suppose Z is uniruled, covered by a rational curve $f\colon \mathbb{P}^1 \to Z$ through a very general point. Let $Y \to \mathbb{P}^1$ be a desingularization of $\mathbb{P}^1 \times_Z X$; its general fiber is birational to a general fiber of π, hence rationally connected, so by (2) Y is rationally connected. But then every point of a very general fiber X_z lies on a rational curve inside Y dominating $\mathbb{P}^1$ and hence not contained in X_z contradicting the maximality property of the MRC fibration. ∎

As a simple application of GHS, we show that the Mumford conjecture is a consequence of Mori's uniruledness conjecture.

Proposition 3.4 (cf. [Kol96, Prop. 5.7]). *Assume Theorem 3.3 and Mori's uniruledness conjecture ($\kappa(X) = -\infty \iff X$ uniruled) hold in dimension $\leq k$. Then the Mumford conjecture (X rationally connected $\iff H^0(X, (\Omega_X^1)^{\otimes m}) = 0$ for all $m > 0$) also holds in dimension $\leq k$.*

Proof. Suppose X is not rationally connected. We will show that $P_{m,n}(X) \neq 0$ for some $m, n \geq 1$. Let $\mu : X \longrightarrow R(X)$ be the MRC fibration. After resolving the rational map we may assume that we have a morphism $\mu' : X' \to Z$ between smooth projective varieties with $X' \to X$ birational. Moreover, since $P_{m,n}$ is a birational invariant, it suffices to prove the result for X', so we may assume that μ is a morphism. By GHS, Z is not uniruled, and since X is not rationally connected, $\dim Z > 0$. By Mori's conjecture, $h^0\left(Z, \omega_Z^{\otimes m}\right) \neq 0$ for some $m \geq 1$. Pulling back using generic smoothness and the birational invariance of $P_{m,n}$, we have an injection

$$H^0\left(\omega_Z^{\otimes m}\right) \hookrightarrow H^0\left(\left(\Omega_X^{\dim Z}\right)^{\otimes m}\right)$$

so $P_{m,n}(X) \neq 0$ for some $m, n \geq 1$. Indeed, consider the codifferential $d\mu : \mu^*\Omega_Z^1 \to \Omega_X^1$, which will induce

$$\phi : \ \mu^*\omega_Z^{\otimes m} = \left(\mu^* \wedge^n \Omega_Z^1\right)^{\otimes m} \xrightarrow{(\wedge^n d\mu)^{\otimes m}} (\Omega_X^n)^{\otimes m}.$$

We first show it is generically non-zero and injective. Indeed, by generic smoothness, we can find an open subset $V \subset X$ on which

$$0 \to \mu^*\Omega_Z^1 \xrightarrow{d\mu} \Omega_X^1 \to \Omega_{X/Z}^1 \to 0$$

is exact; in particular, taking wedge products, $\wedge^n d\mu$ is injective on V, so $\phi|_V \neq 0$. We now claim that $\ker\phi = 0$. Indeed, $\ker\phi$ is a coherent subsheaf of the line bundle $\mu^*\omega_Z^{\otimes m}$, hence torsion-free of rank 0 or 1; since ϕ is generically injective, its rank is 0, and a torsion-free sheaf of rank 0 vanishes. Hence ϕ is injective everywhere. So that we have

$$H^0(X, \mu^*\omega_Z^{\otimes m}) \hookrightarrow H^0(X, (\Omega_X^n)^{\otimes m}),$$

precompose with the injection $H^0(Z, \omega_Z^{\otimes m}) \hookrightarrow H^0(X, \mu^*\omega_Z^{\otimes m})$ gives

$$H^0(Z, \omega_Z^{\otimes m}) \hookrightarrow H^0(X, (\Omega_X^n)^{\otimes m}),$$

■

4 Projectivity of the MRC Fibration

Compared to the Albanese map and the Iitaka fibration, the MRC fibration enjoys a striking extra rigidity: it is projective, even when the ambient variety is only compact Kähler. This is what makes it the natural staging ground for exporting projective minimal model program technology to the Kähler category. The starting point is the vanishing condition we proved in Section 1, which can be used in the Kodaira projectivity criterion.

We next prove Claudon–Höring's projectivity criterion. Using it, we show that the MRC fibration admits a projective model.

Theorem 4.1. *Let $f : X \to Y$ be a fibration between normal compact KÃ¤hler spaces. Assume that X has strongly $\mathbb{Q}$-factorial klt singularities. Assume one of the following:*

- *The normal space Y has klt singularities and the natural map $f^* : H^0\left(Y, \Omega_Y^{[2]}\right) \longrightarrow H^0\left(X, \Omega_X^{[2]}\right)$ is an isomorphism.*
- *The morphism f is Moishezon.*

Then f is projective.

Proof. We prove (1) in the smooth case, which is all we need. Assume X, Y smooth. Set $d := \dim X - \dim Y$ (the relative dimension) and fix a KÃ¤hler class ω on X. For any class $\eta \in H^2(X, \mathbb{Q})$, the first hypothesis forces the $(2,0)$-part to come from Y: the Hodge decomposition reads

$$\eta = f^*(\gamma) + \beta + f^*(\bar{\gamma})$$

with $\gamma \in H^0(Y, \Omega_Y^2) = H^{2,0}(Y)$ and β of type $(1,1)$.

In particular, restricting to any fibre X_y, the pullback terms vanish, so $\eta|_{X_y} = \beta|_{X_y}$ is of type $(1,1)$. We choose η sufficiently close to ω so that its $(1,1)$-part β is still a relative Kähler class.

Replace η by $\tilde{\eta}$ with $\tilde{\eta} - \eta \in f^*H^2(Y, \mathbb{R})$ and $\tilde{\eta}$ of type $(1,1)$. Granting this: $\tilde{\eta}$ is a rational $(1,1)$-class, so by the Lefschetz (1,1)-theorem, for $m \in \mathbb{N}$ sufficiently divisible $m\tilde{\eta} = c_1(L)$ for some $L \in \mathrm{Pic}(X)$. Since $\tilde{\eta}$ differs from η by a pullback and β was relatively KÃ¤hler, $c_1(L)$ is relatively KÃ¤hler, as a line bundle hence it's relative ample. Hence f is projective.

To construct such $\tilde{\eta}$, we expand

$$\eta^d = (\beta + f^*(\gamma + \bar{\gamma}))^d,$$

by projection formula then

$$f_*(\eta^d) = f_*(\beta^d),\ f_*(\eta^{d+1}) = f_*(\beta^{d+1}) + (d+1)\, f_*(\eta^d)\, (\gamma + \bar{\gamma})\text{[a]}.$$

Observe that

$$f_*(\eta^d) = \int_F \beta^d \in H^0(Y, \mathbb{Q})$$

is a positive rational number — positive because $\beta|_F$ is KÃ¤hler on the general fibre F (so $\int_F \beta^d > 0$), and rational as just noted. We now define

$$\eta_Y := \frac{1}{(d+1)\, f_*(\eta^d)} f_*(\eta^{d+1}) \in H^2(Y,\mathbb{Q}), \qquad \tilde{\eta} := \eta - f^*(\eta_Y) \in H^2(X,\mathbb{Q}),$$

then we can see $\tilde{\eta}$ is (1,1)-type. ∎

[a] here we use f_* being a morphism of Hodge structures of bidegree $(-d,-d)$: a degree-$2k$ class pushes to degree $2k-2d$

Corollary 4.2 ([CH24]). *Let $f : X \to Y$ be a fibration between normal compact KÃ¤hler spaces with klt singularities. Assume that X is strongly $\mathbb{Q}$-factorial.*

- *If $R^1 f_*\mathcal{O}_X = R^2 f_*\mathcal{O}_X = 0$ then f is projective.*
- *If $h^i(F,\mathcal{O}_F) = 0$ on general fiber for all $i > 0$ then f is projective.*
- *If f is bimeromorphic, then f is projective.*

Proof. Since $R^1 f_*\mathcal{O}_X = R^2 f_*\mathcal{O}_X = 0$, the Leray spectral sequence shows that $H^2(X,\mathcal{O}_X) \simeq H^2(Y,\mathcal{O}_Y)$. By Hodge symmetry for klt varieties,

$$H^2(X,\mathcal{O}_X) \cong H^0(X,\Omega_X^{[2]}),$$

where we take reflexive power in the isomorphism above. So that we have the commutative diagram

$$\begin{tikzcd} H^0(Y,\Omega_Y^{[2]}) \arrow[r] \arrow[d, "\simeq"'] & H^0(X,\Omega_X^{[2]}) \arrow[d, "\simeq"] \\ H^2(Y,\mathcal{O}_Y) \arrow[r] & H^2(X,\mathcal{O}_X) \end{tikzcd}$$

Since the bottom row is an isomorphism, so is the top row. In particular, we can apply the previous theorem to conclude that f is projective. ∎

Theorem 4.3 (cf. [CH24, Thm. 1.2]). *Let X be a normal compact Kähler space with klt singularities. Then there is a model of the MRC fibration of X that is a projective morphism.*

Proof. Let $f : X \dashrightarrow Y$ be the MRC fibration, almost holomorphic with rationally chain connected general fiber F. Choose a modification $Y' \to Y$ with Y' compact Kähler klt, and a resolution of indeterminacy $X' \to Y'$ with X' compact Kähler with strongly $\mathbb{Q}$-factorial klt singularities (e.g. X' smooth), fitting into

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, dashed, "f"] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

Since the general fiber of f' is bimeromorphic to the general fiber of $X \dashrightarrow Y$, it is a rationally connected manifold. By the vanishing we proved, $h^i(F', \mathcal{O}_{F'}) = 0$ for all $i > 0$. Applying the previous theorem to f' then gives that $f' \colon X' \to Y'$ is projective. ∎

Corollary 4.4 ([NW25, Lem. 6.1]). *Let X be a compact Kähler manifold. If the MRC quotient $R(X)$ of X is projective, then X is projective.*

[NW25] gave a proof using Campana's algebraic connectedness; perhaps the result is now an immediate consequence of [CH24].

5 Applications of the MRC fibrations

We end this note by presenting several applications of the MRC fibration.

5.1 Kollár–Miyaoka–Mori's theorem on families of rationally connected varieties

Rationally connected varieties enjoy nice deformation behavior, first proved by Kollár–Miyaoka–Mori. We prove it here as an application of the MRC fibration.

Theorem 5.1. *Let $f : X \to S$ be a proper and equidimensional morphism. There are countably many closed subvarieties $S_i \subset S$ such that X_s is rationally chain connected iff $s \in \cup S_i$.*

Proof. Let $R/S \subset \mathrm{Chow}(X/S)$ be the closed subset parametrizing connected 1-cycles with rational components[a]. We decompose $R/S = \bigcup R_i$ into its (countably many) irreducible components, with universal families $g_i : U_i \to R_i$. Let

$$Z_i \subset X \times_S X$$

be the image of $U_i \times_{R_i} U_i$. Since cycle map $U_i \to X$ is proper, the image is closed. Hence

$$\big(U_i \times \{s\}\big) \times_{R_i} \big(U_i \times \{s\}\big) = \big(U_i \times_{R_i} U_i\big)_s \xrightarrow{u_s} X_s \times X_s \text{ is dominant } \iff Z_i \times \{s\} = X_s \times X_s,$$

i.e. the image is dominant if and only if it is all of $X_s \times X_s$. We then set

$$S_i = \{\, s \in S \mid Z_i \times \{s\} = X_s \times X_s \,\}.$$

This is a closed locus, by upper semicontinuity of fibre dimension applied to $\{s : \dim(Z_i)_s \geq 2d\}$. Since $f : X \to S$ is flat, $X_s \times X_s$ is equidimensional of dimension $2d$, so the condition defining S_i is equivalent to $\dim(Z_i)_s \geq 2d$. By construction, $\bigcup_i S_i$ is precisely the rationally chain connected locus. ■

[a] The closedness was proved by Kollár.

We next upgrade the locus from analytically meager to analytically closed.

Theorem 5.2 ([Kol96, Thm. 3.11])**.** *Let $g : X \to S$ be a proper smooth morphism, and S is connected. Assume that X_0 is rationally connected. Then X_u is rationally connected for all $u \in S$.*[a]

[a] The result fails in the singular case: a smooth family of cubic surfaces (Fano, hence rationally connected) can degenerate to the cone over an elliptic curve (which has lc, non-klt singularities), which is only rationally chain connected.

Proof. The proof uses an open–closed argument. We first show that if X_0 admits a very free rational curve, then so does X_u for u in some open neighborhood $U \subset S$ of 0. We then show that the locus is also closed for a smooth family.

Step 1. (Euclidean) openness of the locus. The proof is very similar to Fujiki's proof of the deformation-openness of uniruledness.

Step 2. From analytically meager to analytically closed. By the previous result, there are countably many closed $S_i \subset S$ with X_s RCC iff $s \in L = \bigcup S_i$. By a standard Noetherian induction, we may assume S is irreducible and L is dense in S (replacing S by $\bar{L}$); it remains to prove closedness in this setting.

We claim that, to prove closedness, it suffices to show L contains a dense open subset. Indeed, L is analytically meager, so if every component S_i were a proper closed subset, L could not be dense, by the Baire category theorem. We now take the relative MRC fibration.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & R(X/S) \arrow[dl] \\ & S & \end{tikzcd}$$

Here there exists a dense open subset $S^\circ \subset S$ such that for $s \in S^\circ$, $\pi_s : X_s \to Z_s$ is the MRC fibration of X_s. For the proper morphism $g : R(X/S) \to S$, the function $s \mapsto \dim Z_s$ is upper semicontinuous, so $\{s : \dim Z_s \leq 0\}$ is Zariski open. We claim

$$L \cap S^\circ = \{s \in S^\circ : \dim Z_s = 0\},$$

which is therefore Zariski open. The equality holds because, for the MRC fibration, the total space is rationally connected if and only if the MRC quotient is 0-dimensional. Hence L contains a dense open subset. ■

5.2 Hacon–Păun's Cone theorem for Kähler varieties

We now give another application of the MRC fibration: Hacon–Păun's cone theorem for Kähler varieties. We only emphasize the MRC reduction step and leave the rest of the proof aside. Recall that we already showed the MRC fibration is projective, so its fibers are projective varieties. This lets us apply the usual cone theorem on the fibers, turning a transcendental question on a Kähler variety into an honest projective cone-theorem question on a rationally connected fiber.

Theorem 5.3 (Cone theorem for generalized Kähler pairs cf. [HP24, Thm. 0.5])**.** *Assume the BDPP conjecture in dimension n. Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n and $(X, B+\boldsymbol{\beta})$ a generalized klt pair. If $K_X + B + \beta_X$ is pseudo-effective and ω is a Kähler class such that $\alpha = [K_X + B + \beta + \omega]$ is big and nef but not Kähler, then there are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ with*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i], \qquad 0 < -(K_X + B + \boldsymbol{\beta}_X)\cdot\Gamma_i \leq 2n.$$

If moreover $B + \boldsymbol{\beta}_X$ (or $K_X + B + \boldsymbol{\beta}_X$) is big, then I is finite.

Proof. Set $\alpha = [K_X + B + \boldsymbol{\beta} + \omega]$ for a Kähler class ω; we sketch how a α-trivial rational curve is produced (the full cone theorem is then assembled from these curves exactly as in the classical projective case, via bend-and-break).

Since α is not Kähler, its non-Kähler locus $\mathrm{Null}(\alpha)$ has a positive-dimensional component; take Z maximal among these irreducible components. Nakamaye's theorem realizes $\mathrm{Null}(\alpha)$ as the Lelong super-level set of a current in α with analytic singularities, and a log-resolution/subadjunction argument (Kawamata subadjunction for generalized Kähler pairs) produces a generalized pair structure $(Z^\nu, B_{Z^\nu} + \gamma_{Z^\nu})$ on the normalization of Z with

$$K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu} \equiv (c+1)\alpha_{Z^\nu} - \omega_{Z^\nu}$$

for certain log canonical threshold c. Since α_{Z^ν} is nef with $\nu_{\mathrm{num}}(\alpha_{Z^\nu}) < \dim Z$ (this numerical dimension bound is because Z is the null locus of α), intersecting both sides against $\alpha^k_{Z^\nu}\cdot\omega^{\dim Z-k-1}_{Z^\nu}$ shows the left side is strictly negative; since $B_{Z^\nu}\geq 0$ and γ_{Z^ν} is pseudo-effective, this forces K_{Z^ν} and hence $K_{Z'}$ for any resolution $Z' \to Z^\nu$ to be not pseudo-effective.

Now we can apply the MRC machine. Because $K_{Z'}$ is not pseudo-effective, the MRC fibration $Z' \dashrightarrow Y$ of Theorem 2.2 is non-trivial, i.e. $\dim Y < \dim Z'$. Let F be its general fiber, which is positive-dimensional and rationally connected. By Theorem 1.4, $H^0(F, \Omega^2_F) = 0$, i.e. $h^{2,0}(F) = 0$; by Kodaira's projectivity criterion, a compact Kähler manifold with $h^{2,0} = 0$ is projective. So although Z (hence F) may a priori be a transcendental, purely Kähler object, the MRC fibration automatically hands us a projective fiber — exactly the vanishing statement that drove the projectivity criterion of §4 is doing the work here.

A further intersection-number computation shows $\alpha_F := \alpha|_F$ is nef but *not big* on the now-projective variety F. The classical cone theorem, applied to the projective klt pair $(F, (1-\epsilon)\omega_F)$, produces finitely many $\left(K_F + (1-\epsilon)\omega_F\right)$-negative extremal rays; since α_F is nef but not big, at least one such ray R must be α_F-trivial, and the induced contraction $F \to \bar{F}$ is of fiber type, so F is covered by α_F-trivial rational curves C. Since F is projective, Mori's bend-and-break bounds their length: $0 < -K_F \cdot C \leq 2 \dim F$.

Letting $\epsilon \to 0$ transports the length bound from F up through Z' to α-trivial rational curves $C \subset X$ with $0 < -(K_X + B + \boldsymbol{\beta}_X) \cdot C = \omega \cdot C \leq 2 \dim X$, and the full cone theorem follows by the standard argument. ∎

6 Further Directions

We end this note by pointing out some further research directions.

Remark 6.1. [**CRT**] proved, for a smooth proper morphism over a smooth connected curve, that if the general fibers are rational then every fiber is rational, under the additional assumption that the Kodaira dimension of the special fiber is non-negative, using the fiberwise bimeromorphic criterion they developed. This can be viewed as a generalization of [**KT**] to the complex-analytic setting.

Remark 6.2. For varieties with nef anti-canonical divisor, the expected picture is exactly what our tools suggest: the MRC fibration $f : X \to Y$ separates X into a rationally connected part, carrying all the positivity, over a base Y with $c_1(Y) = 0$. The remarkable recent discovery is that this fibration is *locally constant*, proved in the projective case by the recent work of Campana–Cao–Matsumura [CCM21], Juanyong Wang [Wan22], and Matsumura–Wang [MW23], and in the Kähler case by Matsumura–Wang–Wu–Zhang [MWWZ25]. The same structure emerges from semi-positive holomorphic sectional curvature, now over a finite étale quotient of a torus [Mat25]. The technical heart of their proofs is the numerical flatness (positivity) criterion for direct images of [PaT18] and [HPS18].

Remark 6.3. Recall that earlier we proved: if T_X is nef on a complex projective manifold X, then, up to a finite étale cover, X fibers over an abelian variety with Fano fibers. Demailly–Peternell–Schneider also gave a Fano criterion: a rationally connected manifold with T_X nef is Fano. It is conjectured that similar results hold when T_X nef is replaced by weaker conditions.

Remark 6.4 (Campana–Păun's rational connectedness of foliations)**.** [**CP19**; **CP25**] proved the following. Let X be a compact KÃ¤hler manifold and $\mathcal{F} \subset T_X$ a holomorphic foliation. If $\mu_{\alpha,\min}(\mathcal{F}) > 0$ for some $\alpha \in \operatorname{Mov}(X)$, then $\mathcal{F}$ is algebraic and its leaves are rationally connected. Their proof uses the relative MRC fibration. They first show that the foliation is algebraic, so that it comes from an actual fibration $X \to Y$; then, to prove rational connectedness of the leaves, it is enough to prove it for the fibers.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z \arrow[d] \\ & Y \end{tikzcd}$$

The general fiber F of $Z \to Y$ is then the MRC quotient, which by GHS is not uniruled; they show that this contradicts the positivity condition $\mu_{\alpha,\min}(\mathcal{F}) > 0$. Hence F is a point, and $\mathcal{F}$ is a rationally connected foliation.

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