Varities with nef anticanonical divisors Spring 2026

Note IV.0 — Overview

Yi Li

The aim of this note is to give an overview of the problem for varieties with nef anti-canonical divisor. The main results that we will prove are the following:

Theorem 1. Let (X, B) be a klt Kähler pair with B an effective $\mathbb{Q}$-divisor B such that the anti-log canonical $\mathbb{Q}$-line bundle $-(K_X + B)$ is nef. Then, there exists a holomorphic (MRC) fibration $f : X \to Y$ with the following properties:
(1) $f : X \to Y$ is a locally constant fibration with respect to the pair (X, B);
(2) the fiber F is a rationally connected manifold;
(3) the base Y is a compact Kähler manifold with $c_1(Y) = 0$.

We will prove that the Albanese fibration is locally constant for projective Calabi-Yau generalized pairs.

Theorem 2. Let (X, B) be a klt Kähler pair, such that $-(K_X + B)$ is nef. Assume that the Albanese map $\psi : X \to Alb(X)$ is projective (which holds when X is projective), then ψ is locally constant.

We will then show some consequences of the structure theorem including: Beauville-Bogomolov-Yau decomposition for klt generalized pairs, Hacon-McKernan inequality of Kodaira dimension, generic nefness results for varieties with nef anti-canonical divisor, algebraic approximation for pairs with nef anti-canonical divisors, etc.

We will only highlight some key points of the proofs and leave the details to the subsequent notes. The major references of this note are: [CH19],[Cao19], [CCM21], [Wan22], [NW25],[iMW23], [iMWWZ25].

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1 Criteria for a Fibration to be Locally Constant

We first define the locally constant fibration:

Definition 3. Let $f : X \to Y$ be a fiber space between normal analytic varieties and $(X, B + \beta)$ a generalized pair, then we say that f is a locally constant fibration if
(1) f is an analytic fiber bundle with fiber F,
(2) every component of B is f-horizontal,
(3) there is a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$, Δ_F a $\pi_1(Y)$-invariant divisor such that $(X, B) \cong (Y_{\text{univ}} \times F, \mathrm{pr}_2^* \Delta_F) / \pi_1(Y)$ and $p_X^* \boldsymbol{\beta}_X \cong \mathrm{pr}_2^* \boldsymbol{\beta}_F$.

To summarize, we have the following diagram:

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times_Y Y^{univ} \cong (F \times Y^{univ}) \arrow[l, "p_X"'] \arrow[r, "pr_2"] \arrow[d, "pr_1"] & F \\
Y & Y^{univ} \arrow[l, "p_Y"] &
\end{tikzcd}$$

Here are several remarks about locally constant fibrations:

Remark 4.
(1) A locally constant fibration is locally trivial by definition. The converse is not true in general: e.g. consider the projective bundle $\mathbb{P}(E) \to Y$; it is locally constant when E is flat (using the Riemann–Hilbert correspondence). Without the flatness condition on E, it is not clear that a projective bundle is locally constant.
(2) A locally constant fibration induces a splitting of the tangent sheaf on the total space. There is a natural splitting of the tangent sheaf

$$T_{F \times Y^{univ}} = pr_1^* T_{Y^{univ}} \oplus pr_2^* T_F,$$

and this natural splitting induces a splitting of the tangent sheaf T_X into foliations.

The following criterion will be repeatedly used:

Theorem 5 ([iMW23, Proposition 2.5])**.** Let $f : X \to Y$ be a surjective morphism with connected fibers from a normal analytic variety to a smooth variety, and let D be an effective Weil $\mathbb{Q}$-divisor. Assume that f is flat and projective, with an f-relative very ample line bundle L such that the following holds:
(a) $E_m = f_*(mL)$ is a numerically flat bundle,
(b) For every $m \in \mathbb{Z}_{>0}$, the natural morphism

$$\mathrm{Sym}^m E_1 \to E_m$$

is compatible with the flat connection,
(c) For some $k \in \mathbb{Z}_{>0}$ rendering kD a $\mathbb{Z}$-divisor,

$$F_m := f_*(mL - kD)$$

is numerically flat and a subbundle of E_m.

As we see from the theorem above, to check the local constancy of a fibration, it is important to construct the numerically flat direct image sheaves E_m, F_m. The numerical flatness (and weak semipositivity) of these direct images will be the central objects of our focus.

As a corollary of Theorem 5, we have the following local constancy criterion when $-(K_{X/Y}+B)$ is nef (as we can see, the key is to construct some relative ample line bundle L).

Corollary 6 ([iMW23, Theorem 4.1])**.** Let $f : X \to Y$ be a surjective morphism with connected fibers from a normal projective variety X to a smooth Y. Let D be an effective $\mathbb{Q}$-divisor such that:
(a) (X, D) is klt,
(b) $-(K_{X/Y} + D)$ is nef.
Then $f : X \to Y$ is a locally constant fibration with respect to the pair (X, D).

2 Criteria for Numerical Flatness

In order to apply Theorem 5, we need to produce a numerically flat direct image; the following two numerical flatness criteria will be useful. Before that, let us quickly recall the definitions of weakly positively curved and pseudo-effective torsion-free sheaves.

Definition 7 (weakly positively curved, pseudo-effectiveness)**.** Let $\mathcal{E}$ be a torsion-free sheaf and ω be a positive $(1,1)$-form on a compact normal analytic variety X admitting a local potential function.
(1) $\mathcal{E}$ is said to be weakly positively curved if $\mathcal{E}$ admits singular Hermitian metrics $\{g_\varepsilon\}_{\varepsilon>0}$ such that $\sqrt{-1}\Theta_{g_\varepsilon} \succeq -\varepsilon\omega \otimes \mathrm{id}$ on X.
(2) $\mathcal{E}$ is said to be pseudo-effective if, for any $m \in \mathbb{Z}_{>0}$, there exists a singular Hermitian metric h_m on $(\mathrm{Sym}^m \mathcal{E})^{**}$ such that $\sqrt{-1}\Theta_{h_m} \succeq -\omega \otimes \mathrm{id}$ on X.
(3) $\mathcal{E}$ is said to be nef if the Serre line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ is nef.

Remark 8. The curvature is not well defined for torsion-free sheaves $\mathcal{E}$; here $\{g_\varepsilon\}_{\varepsilon>0}$ such that $\sqrt{-1}\Theta_{g_\varepsilon} \succeq -\varepsilon\omega \otimes \mathrm{id}$ means the local function $\log|e|_{g^*} - f$ is plurisubharmonic (psh) on $X_{\mathrm{reg}} \cap X_{\mathcal{E}}$ for any (holomorphic) local section e of $\mathcal{E}^*$.

Definition 9 (Numerical Flatness)**.** We say a torsion-free coherent sheaf $\mathcal{E}$ is numerically flat if $\mathcal{E}$ is nef and $c_1(\mathcal{E}) = 0$. (One can check that $\mathcal{E}$ is numerically flat if and only if both $\mathcal{E}$ and $\mathcal{E}^*$ are nef.)

We now introduce two numerical flatness criteria that we will use:

Theorem 10 ([iMW23, Proposition 2.6])**.** Let $\mathcal{F}$ be a reflexive sheaf on a compact Kähler manifold Y. If $\mathcal{F}$ is weakly positively curved and $c_1(\mathcal{F}) = 0$, then $\mathcal{F}$ is a numerically flat vector bundle on Y.

Theorem 11 ([iMWWZ25, Theorem 2.1])**.** Let $\mathcal{E}$ be a torsion-free sheaf on a compact Kähler manifold X. If $\mathcal{E}$ is pseudo-effective with $c_1(\mathcal{E}) = 0$, then the reflexive hull $\mathcal{E}^{**}$ is locally free and numerically flat on X.

We will prove these two theorems in Note-IV.1 Numerical Flatness Criteria.

3 Positivity of Direct Images

By Section 2, we can reduce the problem to finding a direct image sheaf with weakly positive curvature. The positivity of such direct images will be the core of the proof of the main result.

We first state two consequences of [PaT18, HPS18], which will serve as the source of the remaining positivity results.

Theorem 12 ([PaT18, HPS18])**.** Let $f\colon X \to Y$ be a (projective) surjective morphism with connected fibers between compact Kähler manifolds. Assume that there exists a line bundle L on X equipped with a singular metric h_L such that $\Theta_L \geq f^*\theta$ for some smooth closed $(1,1)$-form θ on Y.

Suppose that there exists an integer $m \in \mathbb{Z}_{>0}$ such that

$$\mathcal{J}\left(h_L^{1/m}\Big|_{X_y} \right) \simeq \mathcal{O}_{X_y}$$

for general $y \in Y$. Assume further that L is f-big and that N is nef on X. Then

$$f_*\mathcal{O}_X(mK_{X/Y} + N + L)$$

is θ-weakly positively curved.

We next state a second positivity result, again a consequence of [PaT18, HPS18].

Theorem 13 ([iMWWZ25, Proposition 3.1])**.** Let $f\colon X \to Y$ be a fibration between compact Kähler manifolds. Consider the following data:

(a) G is an f-big line bundle on X;

(b) N is a nef line bundle on X;

(c) L is a $\mathbb{Q}$-line bundle admitting a singular Hermitian metric h_L with semi-positive curvature such that
$$\mathcal{J}(h_L)|_{X_y} = \mathcal{O}_{X_y}$$
for a general fiber X_y.

Assume that
$$f_*\mathcal{O}_X\big(mK_{X/Y} + G + mN + mL\big)$$
is nonzero for sufficiently large $m \gg 1$, with mL Cartier. Then the $\mathbb{Q}$-line bundle $K_{X/Y} + N + L$ is pseudo-effective.

If we further assume that G is $f^*\theta$-pseudo-effective, then the direct image sheaf

$$f_*\mathcal{O}_X\left(mK_{X/Y} + G + mN + mL\right)$$

is θ-weakly positively curved for $m \gg 1$ with mL Cartier.

For our purposes, consider the MRC fibration $\psi\colon X \dashrightarrow Y$. Resolving the indeterminacy, we obtain a smooth model $\pi\colon X' \to X$ together with an induced morphism $\phi\colon X' \to Y$. Let $E = \mathrm{Ex}(\pi)$. As we will see in the sequel, we can find a ϕ-big line bundle G on X'.

Using this line bundle G, we construct a direct image

$$\mathcal{V}_m = \phi_* \mathcal{O}_{X'}(L_m),$$

where

$$L_m := m(G + cE) - \frac{m}{r}\,\phi^* \det \phi_* \mathcal{O}_{X'}(G + cE),$$

and $r = \mathrm{rk}(\phi_* \mathcal{O}_{X'}(G + cE))$. We will show that

$$\boxed{\mathcal{E}_m = \pi_*\big(\phi^* \mathcal{V}_m\big) \text{ is weakly positively curved}}$$

and that $c_1(\mathcal{E}_m) = 0$. As mentioned above, we will reduce this to [PaT18, HPS18], namely Theorems 12 and 13. For more details on how to prove these positivity results, please refer to Note-IV.2 Weakly Positively Curved Direct Images.

4 Birational Geometry of the MRC/Albanese Fibration

In this section, we give an overview of the birational geometry of the MRC/Albanese fibration. Let $\psi : X \dashrightarrow Y$ be an almost holomorphic (i.e., the indeterminacy locus does not dominate the base), dominant meromorphic map between compact Kähler manifolds. Assume that Y is non-uniruled, and that $\pi : X' \to X$ is a modification such that X' is a Kähler manifold, with the commutative diagram below.

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "\phi"'] & X \arrow[d, dashed, "\psi"] \\ & Y \end{tikzcd}$$

Let $E = \mathrm{Ex}(\pi)$, and write

$$K_{X'} + B'_+ - B'_- = \pi^*(K_X + B).$$

Let $Y_0 \subset Y$ be the maximal Zariski open subset such that ϕ is flat and, for every divisor $D \subset Y_0$, the pullback $\phi^* D$ is not contained in the exceptional locus of π (this is possible since ψ is almost holomorphic).

Theorem 14 ([iMWWZ25, Proposition 3.11])**.** With the assumptions above, the following hold:

(a) Y is birationally Calabi-Yau, i.e. $\kappa(Y) = 0$, and

$$K_Y \sim_{\mathbb{Q}} N_Y \geq 0$$

such that $\phi^* N_Y$ is π-exceptional;

(b) The boundary divisor B is horizontal with respect to ψ;

(c) $\pi(\phi^{-1}(Y \setminus Y_0))$ has codimension ≥ 2 in X (the indeterminacy locus has codimension 2);

(d) Y_0 satisfies the generalized Liouville property: the open set Y_0 satisfies the *generalized Liouville property* in the following sense: let $(\mathcal{H}_0, \nabla_0)$ be a flat vector bundle on Y_0. If it satisfies the lifting condition $(\star)$ below, then every global section of $\mathcal{H}_0$ is parallel with respect to ∇_0.

Lifting condition $(\star)$: There exists a numerically flat vector bundle $\mathcal{H}$ on X such that

$$(\phi^*\mathcal{H}_0, \phi^*\nabla_0) \simeq (\pi^*\mathcal{H}, \nabla) \quad \text{on } M_0 := \phi^{-1}(Y_0),$$

where $\phi^*\nabla_0$ denotes the pullback connection on $\phi^*\mathcal{H}_0$, and ∇ is the (unique) flat connection on $\pi^*\mathcal{H}$;

(e) ψ is semi-stable in codimension 1 (i.e., the ramification part of the pullback divisor ϕ^*P is π-exceptional).

Later, we will prove that Y is not only birationally Calabi-Yau, but actually Calabi-Yau. See Note-IV.3 Birational Geometry of the MRC/Albanese Fibration for the proof of the theorem above.

5 Splitting of the Tangent Sheaf

Having proved local freeness and weak positive curvature of $\mathcal{E}_m$ defined in the previous section, our next goal is to prove that the MRC fibration induces a splitting of the tangent sheaf $T_X = \mathcal{F} \oplus \mathcal{G}$ into foliations. Before that, let us recall some basic definitions about foliations.

Definition 15 (Foliation, leaf of a foliation)**.** Let X be a normal variety. A (singular) foliation is a saturated subsheaf $\mathscr{F} \subseteq T_X$ which is closed under the Lie bracket, i.e. $[\mathscr{F}, \mathscr{F}] \subseteq \mathscr{F}$. A canonical divisor of $\mathscr{F}$ is a Weil divisor $K_{\mathscr{F}}$ (if it exists) such that $\mathcal{O}(-K_{\mathscr{F}}) \cong \det \mathscr{F}$.

A leaf of $\mathscr{F}$ is the maximal connected, locally closed submanifold $L \subseteq X_{\text{reg}}$ such that $T_L = \mathscr{F}|_L$. We say that a leaf is algebraic if it is open in its Zariski closure.

We say a foliation is algebraically integrable if its general leaf is algebraic.

Theorem 16 ([iMWWZ25, Theorem 4.1])**.** Let (X, B) be a klt normal compact Kähler pair, where B is a $\mathbb{Q}$-divisor such that $-(K_X + B)$ is nef, and assume that $\mathcal{E}_m$ is locally free.

Then the MRC fibration $\psi : X \dashrightarrow Y$ induces a splitting of the tangent sheaf

$$T_X = \mathcal{F} \oplus \mathcal{G},$$

such that the following hold:

(a) $\mathcal{F}$ is algebraically integrable, and the general leaves of the foliation are the fibers of the MRC fibration.

(b) The canonical sheaf of $\mathcal{G}$ satisfies $K_{\mathcal{G}} \sim_{\mathbb{Q}} 0$.

By regularity of the foliation, it induces a locally constant fibration $f\colon X \to Z$ that coincides with the MRC fibration $\psi\colon X \dashrightarrow Y$. This completes the proof of the holomorphicity and local constancy of the MRC fibration. For more details, please refer to Note-IV.4 Splitting of the Tangent Sheaf.

6 Putting It All Together

We can now put everything together. Consider the MRC fibration $\psi\colon X \dashrightarrow Y$. Resolving the indeterminacy, we obtain a smooth model $\pi\colon X' \to X$ together with an induced morphism $\phi\colon X' \to Y$. Let $E = \mathrm{Ex}(\pi)$. As the diagram below shows

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "\phi"'] & X \arrow[d, dashed, "\psi"] \\ & Y \end{tikzcd}$$

Step 1. In the first step, we find a ϕ-big line bundle G and we define

$$L_m := L_{c,m} := m(G + cE) - \frac{m}{r}\phi^* \det \phi_*\mathcal{O}_M(G + cE),$$

where $r := \mathrm{rk}\, \phi_*\mathcal{O}_M(G + cE)$, with $\mathcal{V}_m := \phi_*\mathcal{O}_M(L_m)$. We then show that the direct image

$$\mathcal{E}_m = \pi_{[*]}(\phi^*\mathcal{V}_m)$$

is weakly positively curved with $c_1(\mathcal{E}_m) = 0$. (See Note-IV.2 Weakly positive curved of the direct images)

Step 2. We prove that a weakly positive curved torsion free sheaf with vanishing 1st Chern class is locally free and . (See Note-IV.1 Numerical Flatness Criteria)

Step 3. In the third step, we apply the numerical flatness criterion to show that $\mathcal{E}_m$ is locally free and numerically flat. In this case, we show that the MRC fibration induces a splitting of the tangent sheaf T_X (which then implies that the MRC fibration is holomorphic and is a locally constant fibration). (see Note-IV.4 Splitting of the Tangent Sheaf)

Step 4. we invoke Corollary 6 to show that it is actually locally constant with respect to the pair. (see Note-IV.6 Structure Theorem for klt Kahler Varieties with Nef Anti-canonical Bundle).

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