Abundance for Kähler Threefolds

Kähler MMP

Yi Li

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Big picture

A minimal model has a nef adjoint divisor. Abundance says that this numerical positivity comes from a morphism. The hard part is producing enough sections when the divisor is not big. In dimension three the numerical-dimension-two case exposes the central mechanism: Chern-class inequalities, Euler characteristics, and adjunction together force growth of pluricanonical sections.

1 The theorem and what it adds to the MMP

Throughout this note boundaries are rational and adjoint divisors are rational Cartier; semiampleness means generation of a positive integral multiple. This avoids mixing the line-bundle statement with a transcendental notion of semiampleness.

Theorem 1.1 (Abundance for compact Kähler threefolds)**.** *Let (X, B) be a log canonical compact Kähler threefold pair. If $K_X + B$ is nef, then it is semiample. The terminal boundary-free case was developed in [CHP16]; log abundance follows from [DO24] and the numerical-dimension-two theorem of [DO26].*

The 2026 paper supplies both the orbifold Chern-class inequality and the comparison with ordinary Chern classes needed in the argument. Thus the numerical-dimension-two case should not be summarized as following from an inequality alone.

Proposition 1.2. *Let L be a semiample line bundle on a smooth compact Kähler n-fold. Then $\kappa(L) = \nu(L) = \dim Z$, where $f : X \to Z$ is the contraction defined by a sufficiently divisible power of L.*

Proof. Write $L^m = f^*A$ with A ample on the projective normal space Z and $f_*\mathcal{O}_X = \mathcal{O}_Z$. If $d = \dim Z$, the projection formula identifies $H^0(X, L^{mk})$ with $H^0(Z, A^k)$, whose growth has degree d. Thus $\kappa(L) = d$. For a Kähler form ω on X, the class $f^*c_1(A)^d \cdot [\omega]^{n-d}$ is positive: on the locus where f has rank d, its representing semipositive form has strictly positive wedge with ω^{n-d}. Higher powers of $f^*c_1(A)$ vanish because the base has dimension d. This is exactly $\nu(L) = d$. The case $d = 0$ gives both dimensions zero directly. ∎

This is the numerical meaning of abundance: the dimension predicted by intersections agrees with the dimension visible to sections.

2 Why nonvanishing is a separate theorem

A non-torsion degree-zero line bundle P on an elliptic curve is nef but has no section in any positive power. Indeed, a nonzero section of P^m would have an effective zero divisor of degree zero, hence would be nowhere vanishing and trivialize P^m. This contradicts non-torsion. The adjoint hypothesis in abundance is therefore essential; nefness of an arbitrary line bundle does not suffice.

For nef rational Cartier divisors on smooth compact Kähler spaces, nonvanishing completely handles numerical dimension zero.

Lemma 2.1. *If D is nef with $\nu(D) = 0$ and $mD \sim E \geq 0$ for some positive integer m, then $mD \sim 0$.*

Proof. For a Kähler class ω, numerical dimension zero gives $D \cdot \omega^{n-1} = 0$. If E were nonzero, integration of the strictly positive volume form ω^{n-1} over its prime components would give $E \cdot \omega^{n-1} > 0$. But this number is $mD \cdot \omega^{n-1} = 0$. Hence $E = 0$, as required. ∎

For singular adjoint pairs one uses the corresponding intersection theory and reduction results; the lemma isolates the elementary final step, not the nonvanishing theorem itself.

3 The proof architecture in dimension three

Put $D = K_X + B$. The nonvanishing theorem gives $\kappa(D) \geq 0$. Once $\kappa(D) > 0$, the Iitaka fibration lowers the dimension and the threefold abundance arguments use abundance on curves and surfaces, together with relative MMP and extension or descent of sections.

These reductions are proved in [DO24]. They are stronger than the formal existence of an Iitaka map.

The remaining task is to exclude $\kappa(D) = 0$ when $\nu(D) > 0$. The cases $\nu = 0, 1, 3$ are handled in the earlier work; the case $\nu = 2$ is [DO26, Theorem 1.2]. Its proof separates two situations. If lowering the boundary slightly retains pseudoeffectivity, one prepares a model suitable for Chern-class and cohomological estimates. After disposing of the projective case by projective log abundance, if every such decrease destroys pseudoeffectivity, an adjoint-trivial MMP on the remaining nonprojective threefold produces a Mori fibration over a surface; adjunction and the relative geometry then give a lower-dimensional route. These reductions preserve the Kodaira and numerical dimensions, as checked in Section 7 of that paper.

Here is the cohomological mechanism in the first route, separated from the technical construction of the model. The notation S denotes the effective boundary used after that preparation, not an arbitrary divisor on the original threefold.

Lemma 3.1. *Let L be a line bundle on a compact three-dimensional complex space, and S an effective Cartier divisor. Suppose, for all large n, that $\chi(X, L^n)$ is bounded below, $h^2(X, L^n)$ and $h^2(X, L^n(-S))$ are bounded above, and $h^1(S, L^n|_S) \geq an - b$ with $a > 0$. Then $h^0(X, L^n)$ grows at least linearly, so $\kappa(L) \geq 1$.*

Proof. The restriction sequence gives $H^1(X, L^n) \to H^1(S, L^n|_S) \to H^2(X, L^n(-S))$. Consequently $h^1(X, L^n) \geq an - b - C$ for a constant C. The Euler characteristic identity gives

$$h^0(X, L^n) = \chi(X, L^n) + h^1(X, L^n) - h^2(X, L^n) + h^3(X, L^n).$$

The last term is nonnegative, so the assumed bounds imply $h^0(X, L^n) \geq an - C'$. ∎

In the prepared adjoint setting the same argument uses the coherent ideal sheaf of the boundary when it is only a Weil divisor. The source proves the necessary boundedness statements. Chern-class inequalities control the Euler characteristic; if it already grows, sections grow directly. Otherwise adjunction to the boundary and surface abundance force growth of its first cohomology, and the displayed exact sequence transfers that growth back to X. This is the heart of [DO26, Section 8], with its model construction and cohomological estimates retained as supporting inputs in this draft.

4 Consequences and research directions

Combining threefold minimal-model existence with Theorem 1.1 gives a good model for pseudoeffective rational adjoint threefold pairs in the scope of those existence theorems. The semiample morphism and the exceptional-section comparison then give finite generation of the corresponding canonical ring.

For higher dimensions the instructive questions are which Chern-class estimates survive, how to compare singular and resolved invariants, and how to obtain section growth without projective hyperplane sections. For transcendental classes there is an additional issue: there is no pluricanonical section ring for an arbitrary real class. A proposed analogue must first specify the geometric meaning of its conclusion.

References

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