Analytic BCHM for Projective Morphisms

Kähler MMP

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Big picture

Analytic BCHM supplies the projective relative machinery used throughout the Kähler MMP. Projectivity of the morphism is retained; the base is allowed to be analytic. This chapter records the theorem, the logical structure of Fujino's proof, and the passage from good models to finite generation and flips. The longer coefficient and polytope arguments are kept in the separate detailed companion.

1 The theorem and its analytic neighborhood

Let $\pi : X \to Y$ be projective, and let $W \subset Y$ be compact. Fujino's condition (P) asks that X be normal, Y Stein, W a Stein compact, and $W \cap Z$ have finitely many connected components for every analytic subset Z defined near W. A Stein compact has a fundamental system of Stein neighborhoods. These hypotheses supply the noetherian control needed while shrinking around W.

Theorem 1.1 (Fujino, Theorem 1.8)**.** *In this setting, a klt pair (X, Δ) has a log terminal model over a neighborhood of W if Δ is relatively big and $K_X + \Delta$ is relatively pseudoeffective, or if $K_X + \Delta$ is relatively big. In the latter case a log canonical model exists. For rational adjoint divisors the relative canonical algebra is locally finitely generated; see [Fuj22].*

The finite-generation conclusion is local on the base. It does not assert finite generation of global sections over every noncompact Stein open set. Nor does the theorem construct arbitrary global Kähler contractions: the morphism to which it applies is already projective.

2 The proof in four steps

First, nonvanishing is reduced to a sufficiently general projective fibre, where algebraic BCHM applies. Analytic cohomology and base change show that a relevant direct image is nonzero; Cartan's theorem A supplies sections over the Stein base. For real coefficients one does this simultaneously for countably many rational combinations, then uses rational polyhedral approximation. This is Fujino's promotion lemma, not extension of an arbitrary special-fibre section.

Second, the induction constructs good models from effective adjoint divisors. It raises boundary coefficients until a coefficient reaches one. Adjunction to that component makes lower-dimensional finiteness available; pl-flips and special finiteness control the resulting program. The intermediate "neutral model" must then be shown to contract precisely the appropriate fixed divisors. This discrepancy comparison is necessary to obtain a genuine minimal model.

Third, finiteness of models is an induction on the dimension of a boundary polytope. At a parameter Δ_0 with $K_X + \Delta_0 \sim_{\mathbb{R},T} 0$, a point on the segment to Δ satisfies

$$K_X + ((1-t)\Delta_0 + t\Delta) \sim_{\mathbb{R},T} t(K_X + \Delta), \qquad t > 0.$$

Positive scaling preserves nefness and the discrepancy inequalities defining the model, so one reduces to proper faces. In general one first takes the semiample contraction of a central good model; over its target the central adjoint class becomes trivial. Local finiteness of the relevant nef inequalities then brings the models back to the original base. Compactness is used only after this local construction.

Finally, good models give finite generation, and canonical models give flips. In Fujino's notation the induction is

$$F_{n-1} \Rightarrow A_n, \quad E_{n-1} \Rightarrow B_n, \quad (A_n, B_n) \Rightarrow G_n \Rightarrow (E_n, F_n).$$

Here A denotes pl-flips, B special finiteness, G good models, E finiteness of models, and F finite generation. Nonvanishing is proved independently. The strengthened theorem G permits an intermediate projective base while all shrinkings occur in the original Stein base. That strengthening is what lets the induction close.

3 The two consequences used later

> **Proposition 3.1.** *A good minimal model identifies the canonical algebras of the original pair and its model after taking a sufficiently divisible Veronese. That Veronese algebra is locally finitely generated.*

Proof. On a common resolution the pulled-back adjoint divisor of the original pair equals that of the model plus an effective divisor exceptional over the model. Normality gives $q_*\mathcal{O}(mE) = \mathcal{O}$ for divisible m: sections cannot acquire a pole along a divisor downstairs from an exceptional divisor upstairs. Projection formula therefore identifies the two Veronese algebras, including their multiplication.
On the good model a positive multiple of the adjoint divisor is the pullback of a relatively ample line bundle under its semiample contraction. The relative section algebra of that ample bundle is locally finitely generated, and connected fibres identify it with the Veronese algebra in question. ∎

Passing from the Veronese to the full canonical algebra uses Fujino's truncation lemma. This algebraic step is a cited input here, rather than an assertion that every graded algebra is finite over its Veronese.

For a projective flipping contraction, the flip is the relative log canonical model. Construct it on sufficiently small Stein opens in the target. Uniqueness of the canonical model identifies the local constructions on overlaps; uniqueness also gives the cocycle identity. They therefore glue to the flip. This is the local-to-global content of Fujino's flip-existence argument.

4 How this chapter is used in the series

Once a Kähler argument has produced a projective morphism, this theorem gives the relative birational tools. Producing that morphism, preserving Kählerness, and proving transcendental descent remain the tasks of the contraction, projectivity, and base-point-free chapters. The alternative finite-generation approach of Das–Hacon–Păun uses restriction to coefficient-one components and division by their canonical sections; see [DHP24].

The main research interfaces are uniform control of neighborhoods, the availability of projective relative models, and extension or positivity statements with weaker ample hypotheses. The detailed companion retains expanded arguments and their verification notes; they are not prerequisites for reading the rest of this base series.

References

[DHP24] O. Das, C.D. Hacon, and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68.

[Fuj22] Osamu Fujino, *Minimal model program for projective morphisms between complex analytic spaces*, 2022.