BDPP and the Positivity-to-Curves Principle

Kähler MMP Yi Li

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Big picture

This supplement explains the two positivity-to-curves results used by the series. Projective BDPP is a duality theorem for numerical cones; the compact Kähler uniruledness theorem detects the canonical class analytically. Their conclusions meet at the MMP, but their proofs should not be conflated.

1 The projective duality theorem

For a smooth projective variety X, the pseudoeffective cone in $N^1(X)$ is dual to the closed movable cone in $N_1(X)$ [BDPP13]. The easy inclusion follows by restricting effective divisors to general covering curves and passing to limits; this is proved in the rational-curves chapter.

The converse is a separation argument made quantitative. If a divisor lies outside the pseudoeffective cone, one seeks a movable class on which its intersection is negative. Approximate big perturbations on modifications by an ample moving part plus an effective fixed part. The orthogonality estimate says that the latter contributes negligibly in the limiting intersection with powers of the moving part. Pushing those powers down supplies the required movable classes. Fujita approximation by itself does not prove this converse: orthogonality is the additional estimate that makes separation work.

2 The canonical class in the Kähler setting

Ou's theorem states that a smooth compact Kähler manifold is uniruled exactly when its canonical class is not pseudoeffective [Ou25]. Its proof uses analytic positivity and algebraicity criteria for foliations, then a relative Albanese reduction. The algebraicity step is indispensable: a destabilizing analytic direction is not yet a compact rational curve.

The use of projective families is local to particular stages of the argument. It does not mean the original manifold is projective. For example, $\mathbb{P}^1 \times T$ with T a nonprojective compact torus is uniruled and Kähler but nonprojective. Its negative canonical direction is the first factor.

3 The exact consequence needed by the MMP

On a smooth compact Kähler manifold that is not uniruled, the canonical class is pseudo-effective. For projective canonical singularities this extends by the effective discrepancy comparison on a resolution, proved in the rational-curves chapter. This places the space on the minimal-model side of the canonical dichotomy whenever the relevant existence theorem is available.

For a boundary B, the sign of $K_X + B$ must be examined separately. Moreover uniruledness alone does not identify a chosen extremal ray or produce a contraction. The cone and contraction theorems supply those next steps. The longer original BDPP working documents are retained in this folder for subsequent detailed expansion.

4 Research directions

The natural questions are how to extract degree bounds from analytic positivity arguments and how to make them uniform in families. Any proposed cone duality beyond the projective setting must specify the ambient Bott–Chern spaces and which movable objects are used; testing only algebraic curves may lose transcendental information.

References

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.