Ou's proof of BDPP theorem for compact Kähler manifolds Spring 2025

Note 10 — 17, 02, 2025 (draft version round 1)

Yi Li

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The aim of this note is try to give a complete proof of Ou's BDPP theorem.

Theorem 1 ([Ou25, CP25])**.** A compact Kähler manifold is uniruled if and only if K_X is not pseudo-effective.

Contents

1 Overview of the proof

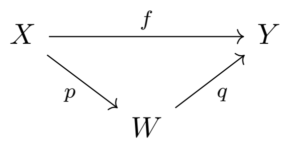
Let us first briefly hightlight some key points of the proof at beginning. We will prove it by contradiction, assume that K_X is not pseudo-effective, then the tangent sheaf T_X will induce a meromorphic fibration

$$X \dashrightarrow Y,$$

[1]**WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

such that $\dim X > \dim Y > 0$ (By Theorem 11). Note that, since we assume X is not projective, there exists a non-zero $\sigma \in H^0(X, \Omega_X^2)$. By Theorem 11, we have $H^0(Y, \Omega_Y^2) \neq 0$ as well.

We then apply the relative Albanese reduction (Theorem 18)



We then apply the pseudo-effectiveness of the adjoint classes in families (Theorem 15) to show that Y is uniruled.

We claim that Y is rationally connected. For if Y is not RC, apply MRC fibration w.r.t Y, we get

$$Y \dashrightarrow V$$

, and did the same thing to

$$X \dashrightarrow V,$$

we deduce V is uniruled and thus contradict to the [GHS03]. So that Y is rationally connected.

In particular, $H^0(Y, \Omega_Y^2) = 0$, which contradiction to $H^0(Y, \Omega_Y^2) \neq 0$ that we claimed at beginning. Hence K_X has to be pseudo-effective.

(Note that technical core of the proof is Theorem 11.)

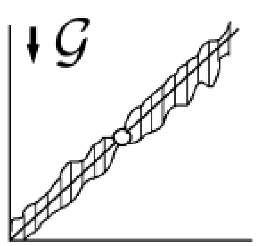
2 Step 1. Algebraic integrability criteria under Kähler setting

2.1 Setting

We will consider a compact Kähler manifold $\hat{X}$, and $C \subset \hat{X}$ be a closed irreducible submanifold. We will consider S_0 be an irreducible, locally closed submanifold containing a Zariski open subset C^o of C.

To understand the setting, let us draw a picture on the specific case that we will use the setting. Given a compact Kähler manifold X, we set $\hat{X} = X \times X$ and C will be the diagonal. We then keep only those regular part of the foliation X^o. Let $Z^o = X^o \times X^o$ and set $C^o = C \cap Z^o$.

We then define a foliation $\mathcal{G} = p_2^{-1}(\mathcal{F}) \cap p_1^{-1}(0)$ on $\hat{X}$, then the local leaf of $\mathcal{G}$ along C^o will defines a locally closed submanifold S^o as the shaded region below shows.



Analytic graph

Our next goal is to give an algebraicity criterion for the foliation which guarantee the $\dim S^o = \dim \overline{S^o}^{Zar}$.

2.2 Comparison of Lelong numbers under birational changes

In this section, we will summarize some Lelong number bound under birational modifications or blow ups. Those bound will be used in finding the quasi-psh function with large restricted Lelong number.

Lemma 2 ([CP25, Lemma 2.2])**.** Let X be a complex manifold, and let $Y \subset X$ be a submanifold. Let $\pi : \widehat{X} \to X$ be the blow-up of Y, and let $E \subset \widehat{X}$ be the exceptional divisor. Let $\widehat{T} \geq 0$ be a positive $(1,1)$-current on $\widehat{X}$ with analytic singularities. Let $\widehat{x} \in E$ and set $x := \pi(\widehat{x})$. Define $T := \pi_*(\widehat{T})$. Then we have

$$C \cdot \nu(T, x) \geq \nu(\widehat{T}, \widehat{x}) - \nu(\widehat{T}, E)$$

for some uniform constant C depending only on X and Y.

Proof. □

Lemma 3 ([CP25, Proposition 4.2])**.** Let X be a compact Kähler manifold and $C \subset X$ be a compact submanifold. Let S_0 be a locally closed submanifold containing C_0, where C_0 is an open subset of C with $\operatorname{codim}_C (C \backslash C_0) \geq 2$.

We fix a point $y_0 \in C_0$. Let ω_X be a Kähler metric on X.

Then there exists some constants K_1 and K_2 depending only on $(\omega_X, y_0, C_0, S_0, X)$ such that, for any ω_X-psh function φ on X, if $\varphi|_{S_0}$ is of algebraic singularities, then we have

$$\nu\left(\varphi|_{S_0}, y_0\right) \leq K_1 \cdot \nu\left(\varphi|_{S_0}, C_0\right) + K_2.$$

Proof. □

2.3 Step 1.1. Finding quasi-psh function with large restricted Lelong number

One of the central lemma that we will use is the following, which allows us to produce some quasi-psh function with large restricted Lelong number. We will mainly follow the purely analytic proof provided by [CP25].

Theorem 4 ([CP25, Proposition 4.4])**.** Let X be a compact Kähler manifold, and let $C \subset X$ be a compact submanifold. Let S_0 be an irreducible, locally closed submanifold containing C_0, where C_0 is a dense open subset of C satisfying $\operatorname{codim}_C (C \backslash C_0) \geq 2$. Let M be the Zariski closure of S_0 in X. Let $\pi : \widehat{X} \to X$ be the blow-up of C in X. Let $\widehat{S}_0$ be the strict transform of S_0, and let D be the exceptional divisor of π. Define $E_0 := \widehat{S}_0 \cap D$. Let $\omega_{\widehat{X}}$ be a Kähler metric on $\widehat{X}$.

If $\dim M > \dim S_0$, then for any $m \in \mathbb{N}$, there exists a $\omega_{\hat{X}}$-psh function φ_m on $\widehat{X}$ such that $\varphi_m|_{\widehat{S}_0}$ has analytic singularities and $\nu\left(\varphi_m|_{\widehat{S}_0}, E_0\right) \geq m$.

PROOF IDEA 5. We will divide the proof into three steps:

Step 1. (Construction of quasi-psh function on the smooth model of $\widehat{M} = \pi_*^{-1}(M)$). To do this, we will take a modification $\widetilde{M} \to \widehat{M}$ and construct a quasi-psh function with large restricted Lelong number along $\widetilde{S}_0$ (More details of the proof will appear in the Proposition 6). The diagram shown as below:

$$\begin{array}{ccccc}
\widetilde{y}_0 \in \widetilde{E}_0 & \hookrightarrow & \widetilde{S}_0 & \hookrightarrow & \widetilde{X} \\
\downarrow p & & \downarrow p & & \downarrow p \\
\widehat{y}_0 \in E_0 & \hookrightarrow & \widehat{S}_0 & \hookrightarrow & \widehat{X} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
y_0 \in C_0 & \hookrightarrow & S_0 & \hookrightarrow & X
\end{array}$$

Step 2. (Using extension theorem extend the quasi-psh function on the whole birational model of $\widehat{X}$). We then use Collins-Tosatti's extension theorem extend the quasi-psh function onto the whole $\widetilde{X}$. Since the extension does not change the part on $\widetilde{M}$, the restricted Lelong number condition does not change.

Step 3. (Push down the associated Kähler current to $\widehat{X}$, trying to bound the Lelong number). Let

$$\widetilde{T}_m = \omega_{\widetilde{X}} + i\partial\bar{\partial}\widetilde{\varphi}_m$$

be the associated Kähler current. We then push it down to

$$\widehat{T}_m = p_*(\widetilde{T}_m), T_m = \pi_* p_*(\widetilde{T}_m).$$

First we can bound

$$\left| \nu\left(\widehat{T}_m\Big|_{\widehat{S}_0}, \widehat{y}_0 \right) - \nu\left(\widetilde{T}_m\Big|_{\widetilde{S}_0}, \widetilde{y}_0 \right) \right| \leq C_1$$

using Lemma **??**. And therefore $\nu\left(\widehat{T}_m\Big|_{\widehat{S}_0}, \widehat{y}_0 \right) \geq m - C_1$. We then divide the problem into two cases:

(a) If $\nu\left(\widehat{T}_m\Big|_{\widehat{S}_0}, E_0 \right) \geq \frac{m}{2}$ for infinitely many $m \in \mathbb{N}^*$, then the proposition is proved.

(b) If $\nu\left(\widehat{T}_m\Big|_{\widehat{S}_0}, E_0 \right) \leq \frac{m}{2}$ for every sufficiently large m, then

$$\nu\left(T_m|_{S_0}, y_0 \right) \geq \frac{1}{K} \cdot \left(\nu\left(\widehat{T}_m\Big|_{\widehat{S}_0}, \widehat{y}_0 \right) - \nu\left(\widehat{T}_m\Big|_{\widehat{S}_0}, E_0 \right) \right) \geq \frac{1}{K} \cdot \left(\frac{m}{2} - C_1 \right)$$

for some uniform constant K independent of m (By Lemma 2). And therefore, by Lemma 3

$$\nu\left(T_m|_{S_0}, C_0 \right) \geq \frac{\nu\left(T_m|_{S_0}, y_0 \right) - K_2}{K_1} \geq \tilde{K}_1 m - \tilde{K}_2.$$

Pull back via p gives the desired quasi-psh function (by Lemma **??**).

Our next goal is to finish the construction of the quasi-psh function on the smooth model $\widetilde{M}$ of $\widehat{M}$ with large restricted Lelong number along $\widetilde{S}$.

Lemma 6 ([CP25, Proposition 4.1])**.** Let X be a compact Kähler manifold, and let ω_X be a Kähler metric on X. Let S_0 be a locally closed submanifold of X such that

$$\dim S_0 < \dim X$$

and the Zariski closure of S_0 is X.

Fix a point $x \in S_0$. Then, for any $\lambda > 0$, we can find a ω_X-psh function φ on X such that φ has analytic singularities and satisfies $\nu\left(\varphi|_{S_0}, x\right) \geq \lambda$.

PROOF IDEA 7. The idea is not hard, we first construct a current with analytic singularity as model and denote it

$$f = \frac{1}{m^{\frac{n-1}{n}}} \log\left(|z_1|^2 + |z_2|^{2m} + \cdots + |z_n|^{2m}\right).$$

We then apply Demailly's mass concentration and Demailly's regularization theorem to find the candiate quasi-psh function with ananlytic singularity which is of the form

$$\varphi = \frac{1}{k} \log \sum |g_i|^2 + O(1).$$

In order to control the restricted Lelong number, only need to give an explicit bound on the vanishing order of each $g_i|_{z_1=0}$ at 0. To do this we use the fact that that each $g_i \in \mathcal{I}(kCf)$. Since we know the local generator of the ideal sheaf well, this will tell us the information on the vanishing order of g_i.

Proof. □

Lemma 8 ([CP25, Lemma 4.5])**.** The same setting as in the proof of Theorem 4. There exists a constant C_1 independent of m such that

$$\left|\nu\left(\widehat{T}\Big|_{\widehat{S}_0}, \widehat{y}_0\right) - \nu\left(\widetilde{T}\Big|_{\widetilde{S}_0}, \widetilde{y}_0\right)\right| \leq C_1.$$

2.4 Step 1.2. Prove the dimension of analytic graphe on the regular part is the same as its Zariski closure

This step, we will mainly follow the approach of [CP25].

Theorem 9 ([CP25, Theorem 1.3])**.** Let X be a compact Kähler manifold, and let $C \subset X$ be a compact submanifold. Let S_0 be an irreducible, locally closed submanifold containing C_0, where C_0 is a dense open subset of C satisfying $\operatorname{codim}_C (C \backslash C_0) \geq 2$. Let M be the Zariski closure of S_0 in X.

If the conormal bundle $\mathcal{N}^*_{C_0/S_0}$ is not pseuod-effective, then $\dim M = \dim S_0$.

PROOF IDEA 10. Assume by contradiction that $\dim M > \dim S_0$. Let us consider the blow up of $\widehat{X} \to X$ along C, with strict transform $\widehat{S}_0$, and D be the exceptional divisor such that $E_0 = D \cap \widehat{S}_0$. First by Theorem 4, for each $m \geq 1$ there exists a $\omega_{\widehat{X}}$-psh function φ_m such that

$$a_m := \nu\left(\varphi_m|_{\widehat{S}_0}, E_0\right) \geq m.$$

Therefore, there exists a sequence of

$$T_m := \omega_{\widehat{X}} + i\partial\bar{\partial}\varphi_m \geq 0$$

positive current such that

$$\boxed{G_m := T_m|_{\widehat{S}_0} - a_m\,[E_0]}$$

are still positive. In particular, this implies

$$c_1\left(G_m|_{E_0}\right) = c_1\left(\omega_{\widehat{X}}\big|_{E_0} + a_m \cdot \mathcal{O}_{\mathbb{P}\left(\mathcal{N}^*_{C_0/S_0}\right)}(1)\right) \quad \text{on } E_0.$$

Which means that $\mathcal{N}^*_{C_0/S_0}$ is pseudo-effective.

2.5 Step 1.3. Finish the proof of the algebraicity criterion for the foliations

We finally reach to point to prove the algebraicity criterion for foliations.

Theorem 11 ([Ou25, Theorem 1.4])**.** Let X be a compact Kähler manifold of dimension n, and let $\mathcal{F}$ be a foliation on X.

(1) If $\mathcal{F}^*$ is non pseudo-effective, then $\mathcal{F}$ is induced by a meromorphic map.

(2) In particular, if the minimal slope satisfies $\mu_{\alpha,\min}(\mathcal{F}) > 0$ for some movable class $\alpha \in H^{n-1,n-1}(X, \mathbb{R})$, then $\mathcal{F}$ is induced by a meromorphic map.

PROOF IDEA 12. To prove this, we will apply Theorem 11 to the analytic graph of the foliation. To be more precise, we set $C' := X$ and $X' = X \times X$, and the regular locus of the foliation $\mathcal{F}$ to be X^o and $C'_0 = C' \cap (X^o \times X^o)$. We define

$$\mathcal{G} = p_2^{-1}(\mathcal{F}) \cap p_1^{-1}(0),$$

to be a foliation on X', so that local leaf of $\mathcal{G}$ will define the locally closed submanifold S'_0. Since we have

$$\mathcal{F}^*|_{X^o} \cong \mathcal{N}^*_{C'_0/S'_0},$$

so that if $\mathcal{F}^*$ is not pseudo-effective, then so will $\mathcal{N}^*_{C'_0/S'_0}$, and hence

$$\dim M = \dim S_0.$$

We finally apply the following Lemma 13, so that $\mathcal{F}$ will induce a meromorphic map.

Lemma 13 ([Ou25, Lemma 7.5])**.** Let X be a compact Kähler manifold and let $\mathcal{F}$ be a foliation on X. Let $X_0 \subseteq X$ be the regular locus of $\mathcal{F}$.

Then $\mathcal{F}$ is induced by a meromorphic map if (and only if) its analytic graph $S_0 \subseteq X_0 \times X_0$ has the same dimension as its Zariski closure in $X \times X$.

(The proof is a bit tricky, I may add the proof later).

Combine those together we have the following corollary.

Theorem 14.

3 Step 2. Pseudo-effectiveness of the adjoint class in families

In the second step, we will present pseudo-effectiveness of adjoint class in family.

Theorem 15 ([Ou25])**.** Let $f : X \to Y$ be a fibration between compact Kähler manifolds. We assume the following conditions.

(1) $H^1(F, \mathcal{O}_F) = \{0\}$ for a general fiber F of f.

(2) K_F is pseudoeffective.

(3) f is smooth over an open subset of Y whose complement is a snc divisor.

(4) There is a Kähler form ω on X and a closed holomorphic 2-form $\tau \in H^0(Y, \Omega_Y^2)$, such that the cohomology class $\{\omega + f^*\tau + f^*\bar{\tau}\}$ belongs to the image of $H^2(X, \mathbb{Z})$.

Then $K_{X/Y}$ is pseudo-effective.

PROOF IDEA 16.

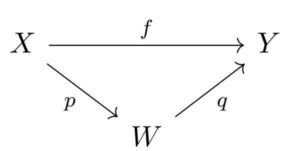
Theorem 17. Let $f : X \to Y$ be a fibration between compact Kähler manifolds. Assume that general fibers F of f have maximal Albanese dimension. Then $\omega_{X/Y}$ is pseudoeffective.

Proof. □

4 Step 3. Relative Albanese reduction

Once obtain the meromorphic fibration Theorem 11, we can factorize it into two parts: (1) The 1st part is a fibration with Albanese dimension 0 or it's of maximal Albanese dimension; (2) General fiber of the 2nd part is not uniruled. (More about relative Albanese map can be found on my note Note-I.2. Albanese map with applications).

Theorem 18 ([Ou25, Proposition 8.6])**.** Let $f : X \to Y$ be a fibration between compact complex analytic varieties, such that $\dim X > \dim Y$ and that X is a Kähler manifold. Then, up to blowing up X, there is a factorization of f into fibrations $p : X \to W$ and $q : W \to Y$, such that the following properties hold.
(1) W is a compact Kähler manifold.
(2) There is a Zariski open subset $W^\circ \subseteq W$, whose complement is a simple normal crossing divisor, such that p is smooth over W°.



(3) **(1st part of the fibration has maximal Albanese dimension or Albanese dimension 0)** Let F be a general fiber of p. Then $\dim F > 0$. In addition, either $H^1(F, \mathcal{O}_F) = \{0\}$ (Albanese dimension 0) or F has maximal Albanese dimension. (Note that since the fiberation is not trivial i.e. $\dim F > 0$ we know that dimension of $p : X \to W$ will drop),

(4) **(2nd part of the fibration general fibers are not uniruled)** General fibers of q are non uniruled. (In particular if Y is not uniruled then W is not uniruled).

PROOF IDEA 19. The idea is repeatly apply the relative Albanese reduction until condition (3) is satisfied. We will mainly focus on condition (3) and condition (4), since (1) and (2) trivially holds after blowing up X and W once (3) and (4) holds.

Assume (3) does not hold. Since X is Kähler, and f is a fibration, the relative Albanese reduction exists by [GPR94, Theorem 3.27]

$$X \xrightarrow{\alpha} \mathrm{Alb}(X/Y) \to Y.$$

We then take the image of the relative Albanese map $\alpha : X \to \mathrm{Alb}(X/Y)$ as $Z = \alpha(X)$. Since (3) does not hold, this implies that

$$\dim X < \dim Z < \dim Y.$$

Since Z is a torus, the general fiber $Z \to Y$ can not be uniruled. Taking a Stein factorization on

$$X \to W \to Z.$$

We can gurantee that the first factor is a contraction morphism.

Since the dimension $\dim X > \dim Z$ (strictly drop), after finite steps of the reductions, (3) and (4) holds.

5 Proof of BDPP conjecture for compact Kähler manifold

Having introduce the previous steps of preparations, we can now finish the proof of Ou's BDPP theorem for compact Kähler manifolds.

Theorem 20 ([Ou25, CP25])**.** A compact Kähler manifold is uniruled iff K_X is not pseudo-effective.

PROOF IDEA 21. First note that a uniruled variety has non-pseudo-effective K_X. The non-trivial part is the converse direction, i.e., if K_X is not pseudo-effective, then X is uniruled. We will prove it by contradiction, and assume that X is not uniruled. We then

Step 1. (Construct meromorphic fibration whose base Y is uniruled). We first show that if K_X is not pseudo-effective, then $\mu_\alpha(T_X) > 0$, we can then apply Theorem 11 to find some meromorphic fibration induced by T_X, and denote it

$$X \dashrightarrow Y.$$

We then apply relative Albanese Theorem 18, and factorize it into

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle q} & \\ & W & \end{array}$$

We then apply the Theorem 15, to show that $K_{X/W}$ is pseudo-effective. Since we assume that K_X is not pseudo-effective, so that K_W is not pseudo-effective. By induction on dimension (since $\dim W < \dim X$), so that W is uniruled, and Theorem 18 again Y is uniruled.

Step 2. (Prove Y is rationally connected). Assume Y is not rationally connected. We then apply the MRC fibration to the base $Y \to V$, (here we resolve the indeterminacy).

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \overset{\bar{f}}{\searrow} & \downarrow g \\ & & V \end{array}$$

By Gabber-Harris-Starr ([GHS03]), V is not uniruled. By Leray spectral sequence arguement

$$H^0(Y, \Omega_Y^2) = H^0(V, \Omega_V^2)$$

Hence we can invoke the Theorem 18 again, and run the same argument as in the Step 1 to show that V is uniruled. This gives a contradiction to $H^0(Y, \Omega_Y^2) \neq 0$.

Proof. □

References

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