BDPP Theorem for Projective Manifolds Spring 2025

Note 10 — 17, 02, 2025 (draft version 1)

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The aim of this note are two:

(1) We will study the rational curves on Kähler varieties ([HP16], [CH20]),

(2) We will give an introduction to the BDPP theorem for projective and Kahler manifolds ([BDPP13],[Ou25]).

Some further discussion and applications can be found in Cone Theorem for Kähler MMP.

Contents

1 Duality Between Varies Positive Cones

In this section, we try to study the duality between varies cone in projective/Kähler settings.

The following theorem shows the duality between pseudo-effective cone and movable cone on the projective manifold.

Lemma 1.1. Let X be a projective manifold. Let $\gamma \in N_1(X)$ be a movable class. Then given any prime divisor E, there exist a representative γ_E such that γ_E intersect E properly and $\gamma \equiv \gamma_E$.

Remark 1.2. I am not pretty sure, if the result is also true for Kähler manifold?

[1]**WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

Proof. □

Theorem 1.3. Let X be a projective manifold, then the pseudo-effective cone is dual to the cone of movable curves

$$\mathcal{E} = \overline{\mathrm{Mov}(X)}^{\vee}.$$

In other words, a divisor is pseudo-effective iff it has non-negative intersection with any movable curves.

Remark 1.4. David [**David19**] proved ...

Remark 1.5. Let us briefly sketch the idea of the proof.

Proof. Let C be a movable curve, By Lemma 1.1, we can choose some C' such that $C \equiv C'$ and C' meets the given pseudo-effective divisor properly. Hence

$$\mathcal{E} \subset \overline{\mathrm{Mov}(X)}^{\vee}.$$

Conversely, if the inclusion is strict then there exist some

$$\xi \in \partial(\mathcal{E}(X)), \quad \xi \in \mathrm{int}\left(\overline{\mathrm{Mov}}(X)^*\right).$$

We want to deduce contradition. Since X is projective, we can find some ample divisor H such that $\xi - \epsilon H$ still in the movable cone. So that

$$\frac{(\xi \cdot C)}{(H \cdot C)} \geq \varepsilon, \quad \forall\ C \in \overline{\mathrm{Mov}(X)}.$$

On the other hand we can apply Fujita approximation to the class $\xi + tH$ for the ample H. And gets

$$\mu_t : X_t \to X$$

such that

$$\mu_t^*(\xi + tH) = A_t + E_t$$

choose $C = \mu_* A_t^{n-1}$, then apply the Asymptotic orthogonality of Fujita approximation to $\xi \cdot C$ and Teissier-Hovanskii inequality to deduce an upper bound

$$\delta_t \geq \frac{\xi \cdot C}{H \cdot C} \geq \epsilon$$

with $\delta_t \to 0$ when $t \to 0$ (here δ_t is a constant depend on the volume of A_t, since $\mathrm{vol}(\xi) = 0$ by Fujita approximation $\mathrm{vol}(A_t) \to 0$ when $t \to 0$).

□

One can generalize the duality theorem to the normal Moishezon space using standard blow up arguement.

Theorem 1.6. Let X be a normal Moishezon space, then the pseudo-effective cone is dual to the movable cone of curves.

Proof. □

Using the duality theorem, we can show that cone of nef curves coinside with the movable cone of curves.

Theorem 1.7. Let X be a normal Moishezon space, then the Batyrev nef cone coinside with the movable cone of curves.

2 Duality between pseudo-effective cone and movable cone

3 BDPP Theorem for Projective manifolds

The projective uniruled manifold is characterized by the pseudo-effectiveness of the canonical bundle.

Lemma 3.1. Given a movable curve C, there exist a covering family $\bigcup_{t\in S} C_t$ contains C, which covers a dense open subset of X. To be more precise, we can find a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & X \\ {\scriptstyle f}\downarrow & & \\ S & & \end{array}$$

with f a fibration, with fibers C_t and ϕ is dominant generic finite morphism, with $\{C_t\}_{t\in S}$ lies the same numerical class.

Proof. □

Theorem 3.2 ([BDPP13, Corollary 0.3])**.** Let X be a projective manifold. Then X is uniruled iff K_X is not pseudo-effectiveness.

Remark 3.3. One direction of the proof is easy, and can be adopted to the Kähler manifold. The converse direction (say K_X is not pseudo-effective) implies uniruled of X is non-trivial, which requires the Mori bend and break technique and the duality between pseudo-effective cone and movable cone.

Remark 3.4. Miyaoka and Mori [**MM86**] proved that a projective manifold is uniruled iff there exist an open subset over which there exist a K_X-negative curve passing through it. For more discussion about Miyaokao-Mori theorem (and varies properties of uniruled manifold) see my Note 15.

Proof. It's sufficient to prove that if K_X is not pseudo-effective, then X is uniruled. By duality of pseudo-effective cone and movable cone, we know that there exists a movable curve such that

$$K_X \cdot C < 0.$$

By Lemma 3.1, we can produce a covering family of K_X-negative irreducible curves using the movable curve C. □

We can generalize the BDPP theorem to the singular case.

Theorem 3.5. Let (X, B) be a $\mathbb{Q}$-factorial log pair. If $K_X + B$ is not pseudo-effective, then X is uniruled.

Remark 3.6. Rational curves on singular space is tricky. See more discussion on my notes note-9 Rational curves on Moishezon space, Kaehler varieties.

Proof. Taking the log resolution

$$f : X' \to X,$$

such that $f^*(K_X+B) = K_{X'}+B'$. Since being uniruled is birational invariant, if X is not uniruled, then so it is X'. Then by the BDPP theorem we just proved, $K_{X'}$ is pseudo-effective, thus K_X is pseudo-effective. Since B is effective, $K_X + B$ is pseudo-effective. □

We can also show the converse direction for canonical singularity.

Theorem 3.7.

The following example indicate that BDPP theorem may fail for singular variety however.

We can characterize the uniruled variety using subsheaf of tangent sheaf

Theorem 3.8. Let X be a projective manifold, $\mathscr{F} \subset T_X$ be a coherent subsheaf such that $\det \mathscr{F}^* \subset T_X$ is not pseudo-effective, then X is uniruled.

Proof. □

4 Several Applications of BDPP Theorem

4.1 Applications of duality of pseudo-effective cone and cone of movable curves

4.2 Producing rational curves using BDPP conjecture

4.3 Cone theorem using BDPP conjecture

[HP24] proved the following cone theorem for klt Kähler pairs. We will discuss detail of the proof in .

Theorem 4.1.