

BDPP reading notes

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## 1 Overview

This aim of this note is to introduce the BDPP theorem for projective [BDPP13] and Kähler manifold [Ou25]. Varies applications of the BDPP theorem are shown.

## 2 Transcendentla cone

On a compact Kähler manifold, there may not have plenty of divisors. To make sense varies positivities, it is necessary to introduce the transcendentla cones.

## 3 Duality between varies cones

The following theorem shows the duality between pseudo-effective cone and movable cone on the projective manifold.

**Lemma 1.** Let  $X$  be a projective manifold. Let  $\gamma \in N_1(X)$  be a movable class. Then given any prime divisor  $E$ , there exist a representative  $\gamma_E$  such that  $\gamma_E$  intersect  $E$  properly and  $\gamma \equiv \gamma_E$ .

**Remark 2.** I am not pretty sure, if the result is also true for Kähler manifold?

*Proof.*

□

**Theorem 3.** Let  $X$  be a projective manifold, then the pseudo-effective cone is dual to the cone of movable curves

$$\mathcal{E} = \overline{\text{Mov}(X)}^\vee.$$

In other words, a divisor is pseudo-effective iff it has non-negative intersection with any movable curves.

**Remark 4.** David [WN19] proved ...

**Remark 5.** Let us briefly sketch the idea of the proof.

*Proof.* Let  $C$  be a movable curve, By Lemma 1, we can choose some  $C'$  such that  $C \equiv C'$  and  $C'$  meets the given pseudo-effective divisor properly. Hence

$$\mathcal{E} \subset \overline{\text{Mov}(X)}^\vee.$$

Conversely, if the inclusion is strict then there exist some

$$\xi \in \partial(\mathcal{E}(X)), \quad \xi \in \text{int}(\overline{\text{Mov}}(X)^*) .$$

We want to deduce contradiction. Since  $X$  is projective, we can find some ample divisor  $H$  such that  $\xi - \epsilon H$  still in the movable cone. So that

$$\frac{(\xi \cdot C)}{(H \cdot C)} \geq \epsilon, \quad \forall C \in \overline{\text{Mov}(X)}.$$

On the other hand we can apply Fujita approximation to the class  $\xi + tH$  for the ample  $H$ . And gets

$$\mu_t : X_t \rightarrow X$$

such that

$$\mu_t^*(\xi + tH) = A_t + E_t$$

choose  $C = \mu_* A_t^{n-1}$ , then apply the Asymptotic orthogonality of Fujita approximation to  $\xi \cdot C$  and Teissier-Hovanskii inequality to deduce an upper bound

$$\delta_t \geq \frac{\xi \cdot C}{H \cdot C} \geq \epsilon$$

with  $\delta_t \rightarrow 0$  when  $t \rightarrow 0$  (here  $\delta_t$  is a constant depend on the volume of  $A_t$ , since  $\text{vol}(\xi) = 0$  by Fujita approximation  $\text{vol}(A_t) \rightarrow 0$  when  $t \rightarrow 0$ ).

□

One can generalize the duality theorem to the normal Moishezon space using standard blow up arguement.

**Theorem 6.** Let  $X$  be a normal Moishezon space, then the pseudo-effective cone is dual to the movable cone of curves.

*Proof.*

□

Using the duality theorem, we can show that cone of nef curves coinside with the movable cone of curves.

**Theorem 7.** Let  $X$  be a normal Moishezon space, then the Batyrev nef cone coinside with the movable cone of curves.

## 4 Characterization of the projective uniruled manifold

The projective uniruled manifold is characterized by the pseudo-effectiveness of the canonical bundle.

**Lemma 8.** Given a movable curve  $C$ , there exist a covering family  $\bigcup_{t \in S} C_t$  contains  $C$ , which covers a dense open subset of  $X$ . To be more precise, we can find a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & X \\ f \downarrow & & \\ S & & \end{array}$$

with  $f$  a fibration, with fibers  $C_t$  and  $\phi$  is dominant generic finite morphism, with  $\{C_t\}_{t \in S}$  lies the same numerical class.

*Proof.* □

**Theorem 9** ([BDPP13, Corollary 0.3]). Let  $X$  be a projective manifold. Then  $X$  is uniruled iff  $K_X$  is not pseudo-effectiveness.

**Remark 10.** One direction of the proof is easy, and can be adopted to the Kähler manifold. The converse direction (say  $K_X$  is not pseudo-effective) implies uniruled of  $X$  is non-trivial, which requires the Mori bend and break technique and the duality between pseudo-effective cone and movable cone.

**Remark 11.** Miyaoka and Mori [MM86] proved that a projective manifold is uniruled iff there exist an open subset over which there exist a  $K_X$ -negative curve passing through it. For more discussion about Miyaokao-Mori theorem (and varies properties of uniruled manifold) see my Note 15.

*Proof.* It's sufficient to prove that if  $K_X$  is not pseudo-effective, then  $X$  is uniruled. By duality of pseudo-effective cone and movable cone, we know that there exists a movable curve such that

$$K_X \cdot C < 0.$$

By Lemma 8, we can produce a covering family of  $K_X$ -negative irreducible curves using the movable curve  $C$ . □

We can generalize the BDPP theorem to the singular case.

**Theorem 12.** Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial log pair. If  $K_X + B$  is not pseudo-effective, then  $X$  is uniruled.

**Remark 13.** Rational curves on singular space is tricky. See more discussion on my notes note-9 Rational curves on Moishezon space, Kaehler varieties.

*Proof.* Taking the log resolution

$$f : X' \rightarrow X,$$

such that  $f^*(K_X + B) = K_{X'} + B'$ . Since being uniruled is birational invariant, if  $X$  is not uniruled, then so it is  $X'$ . Then by the BDPP theorem we just proved,  $K_{X'}$  is pseudo-effective, thus  $K_X$  is pseudo-effective. Since  $B$  is effective,  $K_X + B$  is pseudo-effective. □

We can characterize the uniruled variety using subsheaf of tangent sheaf

**Theorem 14.** Let  $X$  be a projective manifold,  $\mathcal{F} \subset T_X$  be a coherent subsheaf such that  $\det \mathcal{F}^* \subset T_X$  is not pseudo-effective, then  $X$  is uniruled.

*Proof.* □

## 5 Proof of BDPP conjecture for Kähler manifold

Recently, [Ou25] proved the BDPP conjecture for the compact Kähler manifold. In this section, we will briefly introduce the result that he proved.

### 5.1 Algebraic integrability criteria under Kähler setting

### 5.2 Pseudo-effectiveness of the adjoint class

### 5.3 Relative Albanese reduction

### 5.4 Proof of BDPP conjecture for compact Kähler manifold

## 6 Varies applications

### 6.1 Applications of duality of pseudo-effective cone and cone of movable curves

### 6.2 Producing rational curves using BDPP conjecture

### 6.3 Cone theorem using BDPP conjecture

## References

- [BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.
- [MM86] Yoichi Miyaoka and Shigefumi Mori, *A numerical criterion for uniruledness*, Ann. of Math. (2) **124** (1986), no. 1, 65–69.
- [Ou25] Wenhao Ou, *A characterization of uniruled compact kähler manifolds*, 2025.
- [WN19] David Witt Nyström, *Duality between the pseudoeffective and the movable cone on a projective manifold*, J. Amer. Math. Soc. **32** (2019), no. 3, 675–689, With an appendix by Sébastien Boucksom.