Canonical Bundle Formulas and Subadjunction

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Big picture

Adjunction lowers dimension along a divisor. The canonical bundle formula lowers dimension along a fibration. Its two terms record different phenomena: the discriminant measures singular fibres, and the moduli part records variation that remains after those singularities have been accounted for. Subadjunction combines these constructions.

1 Why a boundary alone is not enough

For a contraction $f : (X, B) \to Z$ with $K_X + B \sim_{\mathbb{Q}} f^*L$, we seek a formula on the base. Work first with projective klt-trivial fibrations: the pair is sub-klt over the generic point of Z, and the usual rank-one condition $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) = 1$ is imposed, with the discrepancy b-divisor interpreted on a resolution. It ensures that the construction describes a single adjoint system on the generic fibre. Merely having Kodaira dimension zero on that fibre requires a preliminary reduction to this setting.

For each prime divisor P on the base put

$$t_P = \sup\{t \mid (X, B + t f^*P) \text{ is sub-lc over } \eta_P\}, \quad B_Z = \sum_P (1 - t_P)P, \quad M_Z = L - K_Z - B_Z.$$

The formula $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$ is now a definition of the residual class. Its positivity is a theorem.

Proposition 1.1. *Suppose that over the generic point of P the total space is smooth, $f^*P = \sum_i m_i D_i$, and this divisor together with B has simple normal crossings. Write $b_i = \operatorname{coeff}_{D_i} B$. Then $t_P = \min_i (1 - b_i)/m_i$.*

Proof. In a simple normal crossing subpair, sub-log-canonicity is equivalent to every boundary coefficient being at most one. Adding $t f^*P$ changes the coefficient of D_i to $b_i + t m_i$. The simultaneous inequalities $b_i + t m_i \le 1$ hold precisely for $t \le \min_i (1 - b_i)/m_i$. No additional valuation gives a stricter condition for an SNC subpair. This proves the formula. ∎

For a reduced smooth fibre and zero boundary this gives coefficient zero. For a fibre mD with D smooth generically it gives $1 - 1/m$. Thus the definition is forced by the lc threshold, rather than chosen to resemble the familiar elliptic-surface formula.

2 The moduli part lives on a higher model

Theorem 2.1 (Ambro's moduli theorem, projective klt-trivial setting)**.** *For a projective klt-trivial fibration in the above setting, the moduli b-divisor is b-nef and rational b-Cartier: on a sufficiently high projective birational model $Z' \to Z$, its trace $M_{Z'}$ is nef and its traces on higher models are pullbacks. See [Amb05].*

The proof uses the Hodge-theoretic positivity of the direct image associated with the generically trivial adjoint bundle, after suitable coverings and modification. The threshold calculation separates the boundary singularities from that positivity. This Hodge-theoretic theorem is a supporting input in this base draft.

The higher model matters: M_Z need not itself be nef. On another base model one recomputes the discriminant from the pulled-back fibration; its exceptional coefficients cannot be recovered just by taking strict transforms of B_Z. This is precisely the reason for generalized-pair notation $K_Z + B_Z + \mathbf{M}_Z$.

Proposition 2.2 (The section-ring mechanism)**.** *Suppose $f : X \to Z$ is a contraction of normal spaces, $D \sim_{\mathbb{Q}} f^*L + E$, with D, L, E rational Cartier and $E \ge 0$, and the canonical inclusion $\mathcal{O}_Z \to f_*\mathcal{O}_X(mE)$ is an isomorphism for all sufficiently divisible m. Then the corresponding Veronese section rings of D and L are naturally isomorphic.*

Proof. Choose one integer clearing all denominators and the rational linear equivalence. For every multiple m of it, the projection formula gives $f_*\mathcal{O}_X(mD) = \mathcal{O}_Z(mL) \otimes f_*\mathcal{O}_X(mE) = \mathcal{O}_Z(mL)$. The identifications are induced by the same fixed rational equivalence and canonical sections of mE, so commute with products. Taking global sections proves the assertion. ∎

The direct-image condition is the substantive hypothesis: an arbitrary effective vertical error cannot simply be discarded. Fujino's analytic canonical bundle formula arranges such control after modifications in the relative Iitaka setting; see [Fuj22, Section 21].

3 Subadjunction: restrict first, then descend

For a smooth divisor S on a smooth variety, the residue sequence gives $(K_X + S)|_S = K_S$. If $S+B$ has SNC support and S is not a component of B, restriction gives $(K_X+S+B)|_S = K_S+B|_S$. The singular dlt version replaces $B|_S$ by the different on the normalization of S.

A higher-codimensional lc centre cannot be handled by this elementary restriction alone. On a dlt modification choose a stratum dominating the centre, apply divisorial adjunction along the stratum, and then apply a canonical bundle formula to its morphism to the centre. Stein factorization and any finite descent must be included in this construction. The resulting moduli part explains why subadjunction naturally produces a generalized pair.

> **Theorem 3.1** (Projective subadjunction)**.** *Let (X, B) be a projective lc rational pair, W a minimal lc centre, and A an ample rational divisor. For any rational $\varepsilon > 0$ there is an effective rational divisor B_W such that (W, B_W) is klt and $(K_X + B + \varepsilon A)|_W \sim_{\mathbb{Q}} K_W + B_W$. Here normality of W is part of the subadjunction theorem; see [HK10].*

The small ample term allows the positive contribution to be represented by a general effective boundary while preserving klt singularities. It should not be deleted when quoting this ordinary-pair formulation.

4 The Kähler interface and research directions

The relative projective analytic formula and generalized Kähler formulas in [HP24] are the interfaces used in the later MMP. Their hypotheses include the nature of the morphism and the positivity of the nef data. The projective theorem above does not by itself give an unrestricted formula for every proper Kähler fibration.

The key questions concern positivity and descent of the moduli class, and whether it is semiample on an appropriate model. Nefness and semiampleness are distinct claims. For real transcendental classes, positivity only on curves does not characterize Kähler nefness; an extension of the projective proof must control the actual class, not merely its intersections with algebraic curves.

References

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