Contraction in Kähler MMP reading notes Spring 2024

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1 Overview

The aim of this note is to introduce the contraction theorems in the Kähler minimal model program.

2 Fujiki's blowing down theoren and Grauert contraction theorem

3 Kollár-Mori's extension of contraction theorem

Theorem 1 ([KM92, Theorem 11.4])**.**

4 Das-Hacon's contraction theorem for generalized Kähler PLT pairs

In this section we will introduce the contraction theorem of Das and Hacon.

Theorem 2 ([DH24, Theorem 5.8])**.** Let $(X, S+B+\boldsymbol{\beta})$ be a generalized PLT pair with $\lfloor S+B \rfloor = S$ irreducible, such that the following condition holds

1. S is a $\mathbb{Q}$-Cartier divisor,
2. There exist a contraction morphism $f : S \to T$ such that $-S|_S$ is f-ample,
3. the restriction of the canonical class $-(K_X + S + B + \boldsymbol{\beta})|_S$ is Kähler over T.

Then we can find a (bimeromorphic) contraction morphism $F : X \to Y$, with $F|_S = f$.

Remark 3. Let us briefly sketh the idea of the proof. Compared with the Fujiki blowing down theorem, we do not have Cartier condition on S and we do not have the vanishing of higher direct image (of conormal sheaf) condition in the statement.

Since S is $\mathbf{Q}$-Cartier, there exists a $r \in \mathbf{Z}$ such that rS is Cartier. We first show that there exists on the infinitesimal thickending rS, with positivities preserved under thickening. Second we apply the Serre vanishing and change of index trick showing that the higher direct image of the conormal sheaf vanishs. Then apply the Fujiki blowing down theorem yield the result.

The major difficulty of the proof lies in showing the obstruction of infinitesimal extension vanish. To do this, we need adjunction for the generalized PLT pair and Kawamata-Viehweg vanishing for the complex analytic space.

Proof. □

References

[DH24] Omprokash Das and Christopher Hacon, *On the minimal model program for kähler 3-folds*, 2024.

[KM92] János Kollár and Shigefumi Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992), no. 3, 533–703.