Kähler minimal model program with applications to deformation problems

A PhD Dissertation Defense

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Publications and Preprints

- Christopher D. Hacon, Yi Li and Sheng Rao, *On Pseudo-Effectivity and Volumes of Adjoint Classes in Kähler Families with Projective Central Fiber*, arxiv, 42 pages.

Outline of the study

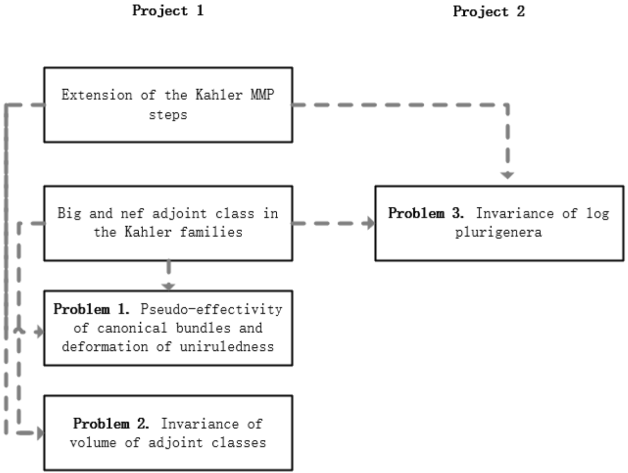

Figure: Outline of the study

Overview

- I. Introduction: Motivation and Background.
- II. Major Tools: Extension of MMP, big and nef adjoint classes in families.
- III. Problem 1: Pseudo-effectivity of canonical bundles and uniruledness under deformations.
- IV. Problem 2: Deformation invariance of volumes of adjoint classes.
- V. Problem 3: (Log) deformation invariance of plurigenera.
- VI. Conclusion: Main Contributions.

Moitivation

Conjecture (Siu's Conjecture)

Let $\pi : X \to \Delta$ be a smooth family of compact Kähler manifolds over the open unit disk Δ with fiber $X_t = \pi^{-1}(0)$. Then for any positive integer m

$$t \longmapsto h^0\left(X_t, mK_{X_t}\right) := \dim H^0(X_t, mK_{X_t})$$

is independent of t for $t \in \Delta$.

Remarkable Results and Consequences

Theorem (Siu98',Siu02')

Let $X \to \Delta$ be a smooth projective family. Then for any positive integer m

$$t \longmapsto h^0(X_t, mK_{X_t}) = \dim H^0(X_t, mK_{X_t})$$

is independent of t for $t \in \Delta$.

Importance of Siu's conjecture

(1) **MMP Theory:** The extension technique used in the proof is a key ingredient in the proof of the MMP conjecture with general type assumption.

(2) **Bergman type metric:** The Bergman-type metric has been extensively studied in recent years.

(3) **Boundedness Problem:** The invariance of plurigenera and volume has been used in the proof of boundedness results for general type problems.

Research Goal

Using minimal model program techniques to study various deformation problems related to Siu's conjecture. We will study:

- Pseudo-effectivity of K_{X_t} in Kähler families;
- Volume $t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \boldsymbol{\beta}_t)$ in Kähler families;
- Log plurigenera $t \longmapsto P_m(K_{X_t} + B_t)$ in Kähler families.

Background: Minimal model program (MMP)

Given a variety with mild singularities, minimal model program is a procedure to find a "canonical" and "nice" representative of the given variety.

Conjecture (Minimal Model Program)

Let (X, B) be a quasi-projective klt pair.
(1) If $K_X + B$ is pseudo-effective, then every mmp terminates

$$\phi : (X, B) \dashrightarrow (X^{(1)}, B^{(1)}) \dashrightarrow \ldots \dashrightarrow (X', B'),$$

so that $K_{X'} + B'$ is nef;
(2) If $K_X + B$ is not pseudo-effective, then every mmp terminates

$$\phi : (X, B) \dashrightarrow (X^{(1)}, B^{(1)}) \dashrightarrow \ldots \dashrightarrow (X', B'),$$

with a Mori fiber space $\varphi : (X', B') \to Z$.

Backgroud: MMP with general type condition

Minimal model conjecture is completely known in the general type case.

Theorem (Birkar-Cascini-Hacon-Mckernan10')

Let (X, B) be a compact klt pair, and X is quasi-projective. Assume further that B is big .
(1) If $K_X + B$ is pseudo-effective, then every mmp terminates

$$\phi : (X, B) \dashrightarrow (X^{(1)}, B^{(1)}) \dashrightarrow \ldots \dashrightarrow (X', B'),$$

so that $K_{X'} + B'$ is nef;
(2) If $K_X + B$ is not pseudo-effective, then every mmp terminates

$$\phi : (X, B) \dashrightarrow (X^{(1)}, B^{(1)}) \dashrightarrow \ldots \dashrightarrow (X', B'),$$

with a Mori fiber space $\varphi : (X', B') \to Z$.

Background: Kähler minimal model program

The following difficulties arise when generalizing the MMP to the Kähler setting:

- Transcendental base point freeness
- Contraction theorems
- Bend-and-break techniques
- Existence of minimal model and termination problems

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Major Tool 1: Extension of MMP

Theorem (Hacon-L.-Rao25')

Let $g : X \to S$ be a flat, proper contraction morphism from a generalized Kähler pair $(X, B + \boldsymbol{\beta})$. Suppose that $(X_0, B_0 + \boldsymbol{\beta}_0)$ is normal $\mathbb{Q}$-factorial, projective, with canonical singularities, and

$$N(K_{X_0} + B_0 + \boldsymbol{\beta}_0) \wedge B_0 = 0.$$

Then every sequence of $(K_{X_0} + B_0 + \boldsymbol{\beta}_0)$-transcendental MMP-steps

$$X_0 \dashrightarrow X_0^{(1)} \dashrightarrow X_0^{(2)} \dashrightarrow \dots$$

***extends** to a sequence of $(K_X + B + \boldsymbol{\beta}_X)$-negative proper meromorphic maps*

$$X/U \dashrightarrow X^{(1)}/U \dashrightarrow X^{(2)}/U \dashrightarrow \dots,$$

over some open neighborhood $U \subset S$ of 0.

Major Tool 2: Big and nef adjoint class in family

Using the projective transcendental base point freeness, we can prove

Theorem (Hacon-L.-Rao25')

Let $f: X \to S$ be a proper surjective morphism from a normal $\mathbb{Q}$-factorial generalized Kähler pair $(X, B + \boldsymbol{\beta})$ onto a smooth, connected, relatively compact curve S, and ω a Kähler form on X. Fix a point $0 \in S$ and assume that the support of the boundary divisor B does not contain the fiber X_0.
Assume that the restriction to the central fiber $(X_0, B_0 + \boldsymbol{\beta}_0)$ is a projective generalized klt pair, such that $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is nef and big.
Then $K_X + B + \boldsymbol{\beta}_X$ is nef and big over U for some open neighborhood $U \subset S$ of 0.

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Problem 1: Pseudo-effectivity of canonical bundles and uniruledness under deformations.

Theorem (Fujiki–Levine–Ou25')

Let $f\colon X \to S$ be a smooth family from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. If the canonical divisor K_{X_0} is pseudo-effective, then K_{X_t} are pseudo-effective for all $t \in S \setminus \{0\}$.

Theorem (Hacon-L.-Rao25')

Let $f\colon X \to S$ be a family from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Assume that X_0 is projective, and X_t have canonical singularities for all $t \in S$. Then the canonical divisor K_{X_0} is pseudo-effective if and only if K_{X_t} are pseudo-effective for all $t \in S \setminus \{0\}$.

Problem 1: Pseudo-effectivity of canonical bundles and uniruledness under deformations

Theorem (Fujiki–Levine81')

Let $f\colon X \to S$ be a smooth family from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Then X_0 is uniruled if and only if X_t are uniruled for all $t \in S \setminus \{0\}$.

Theorem (Hacon–L.–Rao25')

Let $f\colon X \to S$ be a family from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Assume that X_0 is projective, and X_t have canonical singularities for all $t \in S$. Then X_0 is uniruled if and only if X_t are uniruled for all $t \in S \setminus \{0\}$.

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Problem 2: Deformation invariance of volumes of adjoint classes

Theorem (Hacon–Mckernan–Xu18')

Let $f\colon X \longrightarrow S$ be a projective morphism of smooth varieties. Suppose that (X, Δ) is log canonical and has simple normal crossings over S. Then the volume function

$$\mathrm{vol}\,(X_t, K_{X_t} + \Delta_t)$$

is independent of $t \in S$.

Theorem (Hacon-L.-Rao25')

Let $f\colon X \to S$ be a smooth, proper morphism from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Assume that $(X, B + \boldsymbol{\beta})$ is a generalized klt pair such that (X, B) is log smooth over S and $\boldsymbol{\beta} = \overline{\beta}$ where β is nef over S and X_0 is projective. If $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is big, then the volume function

$$t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \boldsymbol{\beta}_t)$$

is constant on an open neighborhood $U \subset S$ of 0.

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Problem 3: (Log) deformation invariance of plurigenera

Theorem (Kollár21')

Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Fix a point $0 \in S$ and assume that the fiber X_0 is projective, of general type, and with canonical singularities. Then there is an open neighborhood $0 \in U \subset S$ such that the plurigenera of X_s are independent of $s \in U$ for every r.

Theorem (L.-Rao26')

Let $g : X \to S$ be a proper, flat morphism from a Kähler pair (X, B) to a relatively compact, smooth, connected curve S, and assume that (X, B) is log smooth over S. Suppose that (X_0, B_0) is projective with canonical singularities and that $K_{X_0} + B_0$ is big, with $N(K_{X_0} + B_0) \wedge B_0 = 0$. Then there exists an open neighborhood $U \subset S$ of 0 such that for any $m \geq 1$, the m-log plurigenus

$$t \longmapsto h^0(X_t,\ m(K_{X_t} + B_t))$$

is constant for $t \in U \subset S$.

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Conclusion: Main Contributions

1. We proved the deformation stability of pseudo-effectivity of canonical bundles, and we generalized Fujiki–Levine deformation stability of uniruledness to the singular case.
2. We extend the deformation invariance result on log volumes due to Hacon–McKernan–Xu. We show that for a log smooth Kähler family, if $K_{X_0} + B_0 + \beta_0$ on the central fiber is big, then $t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \beta_t)$ is constant.
3. We generalize János Kollár's result on the deformation invariance of plurigenera to the deformation invariance of log plurigenera.

Thank You!