An Overview of the Kähler Minimal Model Program

Kähler MMP Yi Li

1 Overview

The aim of this note is to give a brief survey on recent developments in the Kähler minimal model program. This note is intended as an overview of what is currently known about the Kähler minimal model program, and briefly sketches some of the key ideas that appear along the way.

Throughout, the reader is assumed to be familiar with the projective minimal model program at the level of the cone, contraction and base point free theorems, and with the transcendental objects that replace divisors in the Kähler setting: Bott–Chern cohomology classes and the positivity cones, quasi-plurisubharmonic functions, closed positive currents and Lelong numbers, and generalized pairs. These are set up in the companion notes of this series, to which the relevant sections below refer. Results here are stated rather than proved; where a proof is sketched, it is marked as an idea of proof and the details are deferred to the note of this series devoted to that result.

Contents

1 Overview 1

2 Cone Theorems 2

3 Rational Curves on Kähler Varieties and Ou's BDPP Theorem for Compact Kähler Manifolds 2

4 Contraction Theorems 3

5 Transcendental Base Point Freeness 4

6 Existence of Minimal Models, Good Minimal Models, and Canonical Models 4

7 Kähler Criterion and Projectivity Criterion 6

8 Canonical Bundle Formulas 7

9 Abundance Conjecture for Kähler Pairs 7

10 Several Applications of Kähler minimal model program 7

2 Cone Theorems

The original proof of the cone theorem relies on Mori's bend-and-break technique. Unfortunately, this technique remains unavailable for Kähler manifolds. Using the recent resolution of the BDPP conjecture [Ou25], the cone theorem for Kähler varieties has nonetheless been established.

Theorem 2.1 ([HP24, Theorem 0.5]). *Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B+\boldsymbol{\beta})$ is generalized klt and $K_X+B+\boldsymbol{\beta}_X$ is pseudo-effective. Then the following hold:*
(1) (Cone theorem) There are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X)=\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}+\sum_{i\in I}\mathbb{R}^+[\Gamma_i].$$

(2) (Length of extremal rays) $0<-\left(K_X+B+\boldsymbol{\beta}_X\right)\cdot\Gamma_i\leq 2n$.
(3) (Finiteness) If $B+\boldsymbol{\beta}_X$ or $K_X+B+\boldsymbol{\beta}_X$ is big, then I is finite.

Idea of proof. A key observation is that the maximal-dimension component Z of the null locus $\mathrm{Null}(\alpha)$ is covered by rational curves.

Let us give a remark on the history of cone theorems in the Kähler minimal model program: In [HP16], Höring–Peternell developed a Mori bend-and-break theorem for Kähler threefolds, using which they proved the cone theorem for Kähler threefolds with terminal singularities. Later, [CHP16, DO24] use a similar idea to generalize it to the cone theorem for Kähler threefold dlt pairs. Motivated by [CH20], [HP24] completes the proof of the cone theorem for Kähler klt pairs in any dimension.

3 Rational Curves on Kähler Varieties and Ou's BDPP Theorem for Compact Kähler Manifolds

Rational curves play a central role in birational geometry, which is also the case for the Kähler minimal model program.

Recently, Ou [Ou25] proved the BDPP conjecture for compact Kähler manifolds.

Theorem 3.1 ([Ou25, Theorem 1]). *Let X be a compact Kähler manifold. Then K_X is not pseudo-effective if and only if X is uniruled.*

Mori bend-and-break is a well-established theory for projective varieties. However, to date it is known only for Kähler threefolds.

Theorem 3.2 ([HP16, Corollary 5.7]). *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudo-effective. Let $S_1, \dots, S_k$ be components of the Zariski decomposition of K_X. Let*

$$b := \max\left\{1, -K_X \cdot Z \mid Z \text{ a curve s.t. } Z \subset S_{j,\,sing} \text{ or } Z \subset S_j \cap S_{j'}\right\},$$

and set

$$d := \max\{3, b\}$$

If $C \subset X$ is a curve such that $-K_X \cdot C > d$, then we have

$$[C] = [C_1] + [C_2]$$

with C_1 and C_2 effective 1-cycles (with integer coefficients, need not be irreducible) on X.

4 Contraction Theorems

Hacon–Xie [HLX26a] completes the proof of contraction theorem for Kähler minimal model program. The proof of this result is quite technical; we will devote a great deal of effort in this series to explaining it clearly.

Theorem 4.1 ([HLX26a, Theorem 1.6]). *Let $(X, B + \boldsymbol{\beta})$ be a n-dimensional compact Kähler generalized pair such that $B + \boldsymbol{\beta}_X$ is modified big and $\alpha = K_X + B + \boldsymbol{\beta}_X$ is nef and NQC. Assume that the contraction defined by $[\alpha]|_N$ exists and is Moishezon, where $N = \mathrm{Nklt}(X, B + \boldsymbol{\beta})$ is the closed subspace defined by the multiplier ideal $\mathcal{J}(X, B + \boldsymbol{\beta})$.*
Then the contraction defined by $[\alpha]$ exists and is Moishezon.

Note that for a gklt pair the multiplier ideal is trivial, so $N = \varnothing$ and the hypothesis on $[\alpha]|_N$ is vacuous; the theorem is stated for a possibly non-gklt pair because the induction in [HLX26a] runs over log canonical centres, on which gklt fails.

Historically, for Kähler threefolds Höring–Peternell [HP16] claimed the result for $\mathbb{Q}$-factorial compact Kähler threefolds with terminal singularities. Das–Hacon [DH25, DH24a] gave a complete proof for Kähler threefolds with $\mathbb{Q}$-factorial dlt singularities (see [DH24a, Theorem 1.2]).

When a flipping contraction exists, Das–Hacon proved the following result on the existence of flips for generalized klt Kähler pairs in any dimension.

Theorem 4.2 ([DH24a, Theorem 5.12]). *Let $(X, B + \boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial gdlt pair, where X is a compact analytic variety belonging to Fujiki's class $\mathcal{C}$, and $f : X \to Z$ a flipping contraction. Then the flip of f exists.*

Idea of proof. The idea of the proof is to use the local existence of log canonical models; since these are unique, they glue to the flip.

5 Transcendental Base Point Freeness

The transcendental base-point-freeness is conjectured by Tosatti [Tos18], which has been solved recently by Hacon–Xie [HLX26a].

Theorem 5.1 ([HLX26a, Theorem 1.4]). *Let $(X, B + \boldsymbol{\beta})$ be a compact, n-dimensional, Kähler, gklt pair such that $\alpha = K_X + B + \boldsymbol{\beta}_X$ is nef, and $B + \boldsymbol{\beta}_X$ is modified big. Then α is semiample, more precisely, there exists a Moishezon contraction $f : X \to Y$ such that*
(1) Y is a compact Kähler space, and
(2) $\alpha \equiv f^*\gamma$ for some Kähler class γ on Y.
Moreover, if X is strongly $\mathbb{Q}$-factorial, then f is projective.

Historically, for projective pairs the base-point-free theorem holds; see [DH24b, Theorem 3.5].

6 Existence of Minimal Models, Good Minimal Models, and Canonical Models

In this section, we will talk about the existence of minimal model in Kähler MMP.

6.1 Existence of Minimal Models for Kähler Threefolds

For Kähler threefolds, the result is relatively complete ([DH24a], using Shokurov's difficulty function for termination of the MMP).

Theorem 6.1 ([DH24a, Theorem 1.4]). *Let (X, B) be a dlt pair where X is a $\mathbb{Q}$-factorial compact Kähler threefold. If $K_X + B$ is pseudo-effective, then there exists a finite sequence of*

flips and divisorial contractions

$$\phi : X \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_n$$

such that $K_{X_n} + \phi_* B$ *is nef.*

Later, Das–Hacon–Yáñez proved the result for generalized Kähler threefolds.

Theorem 6.2 ([DHY23, Theorem 1.2]). *Let* $(X, B + \boldsymbol{\beta})$ *be a generalized compact klt Kähler threefold pair.*
(1) If $K_X + B + \boldsymbol{\beta}_X$ *is big, then* $(X, B + \boldsymbol{\beta})$ *has a log terminal model* $f : X \dashrightarrow X^{\mathrm{m}}$ *and a unique log canonical model* $g : X^{\mathrm{m}} \to X^{\mathrm{c}}$.
(2) If $K_X + B + \boldsymbol{\beta}_X$ *is pseudo-effective and* $\boldsymbol{\beta}_X$ *is big, then* $K_X + B + \boldsymbol{\beta}_X$ *has a log terminal model* $f : X \dashrightarrow X^{\mathrm{m}}$, *and there is a contraction* $g : X^{\mathrm{m}} \to Z$ *such that* $f_* \left(K_X + B + \boldsymbol{\beta}_X\right) \equiv g^* \omega_Z$, *where* ω_Z *is a Kähler form on* Z.

6.2 Existence of Minimal Models for Kähler Fourfold Effective Pairs

For effective Kähler fourfold dlt pairs, the existence of minimal models was proved by Das–Hacon–Păun.

Theorem 6.3 ([DHP24, Theorem 1.1]). *Let* (X, B) *be a* $\mathbb{Q}$*-factorial compact Kähler fourfold dlt pair such that* $K_X + B \sim_{\mathbb{Q}} M \geq 0$. *Then* (X, B) *has a log minimal model.*

Theorem 6.4 ([DHP24, Corollary 7.12]). *Let* (X, B) *be a* $\mathbb{Q}$*-factorial compact Kähler plt pair of dimension* 4 *such that* $\kappa(X, K_X + B) \geq 0$. *Then* (X, B) *has a log terminal model.*

Idea of proof. Their proof introduces the Θ function, which appeared in [Bir12].

6.3 Kähler BCHM

The strongest existence result is [HLX26a], Hacon–Xie proved an analog of [BCHM10] in the Kähler setting.

Theorem 6.5 ([HLX26a, Theorem 1.5]). *Let* $(X, B + \boldsymbol{\beta})$ *be a compact, n-dimensional, Kähler, strongly* $\mathbb{Q}$*-factorial, gklt pair such that* $B + \boldsymbol{\beta}_X$ *is big, and* ω *a modified Kähler form such that* $K_X + B + \boldsymbol{\beta}_X + \omega$ *is nef. Then we can run the* $K_X + B + \boldsymbol{\beta}_X$ *minimal model program with scaling of* ω *which terminates with either a good log terminal model or a Mori fiber space.*

6.4 Other existence of (good) minimal model results

Das–Hacon proved the existence of good minimal models for Kähler klt pairs of maximal Albanese dimension.

Theorem 6.6 ([DH24c, Theorem 1]). *Let (X, B) be a compact Kähler klt pair of maximal Albanese dimension. Then (X, B) has a good minimal model.*

Remark 6.7 (Comparison with the projective case). Fujino [Fuj13, Theorem 4.3] proved that projective klt pairs of maximal Albanese dimension admit good minimal models. It is instructive to compare Das–Hacon's result with Fujino's and identify what is new in the Kähler setting.

Recently, Huang proved the existence of good minimal models for klt Kähler pairs when the Albanese map is projective and the general fibers admit good minimal models.

Theorem 6.8 ([Hua25, Theorem 1]). *Let $a_X : X \to A$ be the Albanese morphism of a compact Kähler variety X, where (X, B) is a klt pair. Assume a_X is projective with connected fibers, and let F be the general fiber of a_X. If (F, B_F) has a good minimal model, where $B_F = B|_F$ is the restricted boundary, then (X, B) has a good minimal model.*

7 Kähler Criterion and Projectivity Criterion

Hacon–Li–Xie proved that the Kähler MMP preserves the Kähler condition in arbitrary dimension.

Theorem 7.1 ([HLX26b, Theorem 1.2]). *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt pair, where $B + \boldsymbol{\beta}_X$ is modified big, and X is compact, Kähler, and strongly $\mathbb{Q}$-factorial. If $f : X \to Y$ is a flipping or divisorial contraction associated to an extremal ray Γ, then Y is Kähler.*

There are several interesting criteria for projectivity and Kählerness: Claudon–Höring developed a projectivity criterion for Kähler morphisms.

Theorem 7.2 ([CH24, Theorem 1.1]). *Let $f : X \to Y$ be a fibration (that is, a surjective morphism with connected fibres) between compact Kähler manifolds. Assume one of the following holds:*
(1) The natural map $f^* : H^0\left(Y, \Omega_Y^2\right) \longrightarrow H^0\left(X, \Omega_X^2\right)$ is an isomorphism; or
(2) The morphism f is Moishezon, i.e. there exists a line bundle L on X that is f-big.
Then f is a projective morphism.

8 Canonical Bundle Formulas

The canonical bundle formula for proper morphisms between compact Kähler manifolds remains an open question. Some partial results are known for projective morphisms between compact Kähler varieties [HP24].

Theorem 8.1 ([HP24],[HLX26a, Theorem 2.53]). *Let $f : X \to Z$ be a surjective projective morphism of normal compact Kähler varieties with connected fibers, $(X, B+\boldsymbol{\beta})$ a generalized pair which is gklt over an open subset U of Z, such that $B+\boldsymbol{\beta}_X$ is modified big, $K_X + B + \boldsymbol{\beta}_X \equiv f^*\gamma$ for some $\gamma \in H^{1,1}_{\mathrm{BC}}(Z)$. Then $(Z, \boldsymbol{\Delta}_Z + \boldsymbol{\delta})$ is a generalized pair such that $(U, \boldsymbol{\Delta}|_U + \boldsymbol{\delta})$ is gklt, i.e.*
(1) $\boldsymbol{\Delta}_Z \geq 0$,
(2) $\boldsymbol{\delta}$ is b-nef, that is $\boldsymbol{\delta}$ descends to some model Z' so that $\boldsymbol{\delta}_{Z'}$ is nef,
(3) the coefficients of components of $\boldsymbol{\Delta}_{Z'}$ whose images intersect U are < 1 for any birational model $Z' \to Z$.
Moreover, possibly after replacing $(X, B + \boldsymbol{\beta})$ by a generalized pair $\left(X, B' + \boldsymbol{\beta}'\right)$ where $K_X + B + \boldsymbol{\beta}_X \equiv K_X + B' + \boldsymbol{\beta}'_X$ and $\left(X, B' + \boldsymbol{\beta}'\right)$ satisfies the hypothesis of the theorem, we may assume that $\boldsymbol{\delta}_Z$ is modified big.

9 Abundance Conjecture for Kähler Pairs

Abundance for rational lc pairs on Kähler threefolds follows from the combined work of Das–Ou [DO24, DO25].

Theorem 9.1 ([DO24, Theorem 1.1] + [DO25, Corollary 1.3]). *Let (X, Δ) be an lc pair such that X is a compact Kähler threefold. If $K_X + \Delta$ is nef, then it is semi-ample.*

10 Several Applications of Kähler minimal model program

10.1 Deformation of Positivity in Kähler Families

Hacon–Li–Rao proved the following deformation result for pseudo-effective adjoint classes in families, which was later generalized to Kähler families without assuming that the central fibre is projective.

Theorem 10.1 ([HLR26],[HLX26a, Corollary 7.8]). *Let $f : X \to S$ be a family from a Kähler variety X onto a smooth, connected, and relatively compact curve S. Assume that the fibers X_t have canonical singularities for all $t \in S$. Then K_{X_0} is pseudo-effective if and only if K_{X_t} are pseudo-effective for all $t \in S\backslash\{0\}$, and X_0 is uniruled if and only if X_t are uniruled for all*

$t \in S\backslash\{0\}$.

The proof depend on the following deformation invariance of big and nef adjoint classes.

Theorem 10.2 ([HLR26, Proposition 4.24]). *Let $f : X \to S$ be a proper surjective morphism from a generalized Kähler pair $(X, B + \boldsymbol{\beta})$ onto a smooth, connected, relatively compact curve S. Fix a point $0 \in S$ and assume that the support of the boundary divisor B does not contain the fiber X_0.*
Assume that the restriction to the central fiber $(X_0, B_0 + \boldsymbol{\beta}_0)$ is a generalized klt pair, such that $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is nef and big. Then $K_X + B + \boldsymbol{\beta}_X$ is nef and big over U for some Euclidean open neighbourhood $U \subset S$ of 0.

References

[BCHM10] C. Birkar, P. Cascini, C.D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society **23** (2010), no. 2, 405–468.

[Bir12] Caucher Birkar, *Existence of log canonical flips and a special LMMP*, Publ. Math. Inst. Hautes Études Sci. **115** (2012), 325–368. MR 2929730

[CH20] Junyan. Cao and A. Höring, *Rational curves on compact Kähler manifolds*, J. Differential Geom. **114** (2020), no. 1, 1–39.

[CH24] Benoît Claudon and Andreas Höring, *Projectivity criteria for kähler morphisms*, 2024.

[CHP16] F. Campana, A. Höring, and T. Peternell, *Abundance for Kähler threefolds*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 971–1025.

[DH24a] O. Das and C.D. Hacon, *On the minimal model program for Kähler 3-folds*, 2024.

[DH24b] O. Das and C.D. Hacon, *Transcendental minimal model program for projective varieties*, 2024.

[DH24c] Omprokash Das and Christopher Hacon, *Existence of good minimal models for Kähler varieties of maximal Albanese dimension*, C. R. Math. Acad. Sci. Paris **362** (2024), 83–91. MR 4762189

[DH25] _____, *The log minimal model program for Kähler 3-folds*, J. Differential Geom. **130** (2025), no. 1, 151–207.

[DHP24] O. Das, C.D. Hacon, and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68.

[DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.

[DO24] O. Das and W.-H. Ou, *On the log abundance for compact Kähler threefolds*, Manuscripta Math. **173** (2024), no. 1-2, 341–404.

[DO25] Omprokash Das and Wenhao Ou, *On the log abundance for compact Kähler threefolds ii*, 2025.

[Fuj13] Osamu Fujino, *On maximal Albanese dimensional varieties*, Proc. Japan Acad. Ser. A Math. Sci. **89** (2013), no. 8, 92–95. MR 3127923

[HLR26] Christopher D. Hacon, Yi Li, and Sheng Rao, *On pseudo-effectivity and volumes of adjoint classes in kähler families with projective central fiber*, 2026.

[HLX26a] Christopher Hacon, Yi Li, and Lingyao Xie, *Fujiki class $\mathcal{C}$ varieties and a kähler criterion*, 2026.

[HLX26b] ———, *Fujiki class $\mathcal{C}$ varieties and a kähler criterion*, 2026.

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.

[HP24] C.D. Hacon and M. Păun, *On the canonical bundle formula and adjunction for generalized kaehler pairs*, 2024.

[Hua25] Yu-Ting Huang, *On the existence of good minimal models for kähler varieties with projective albanese map*, 2025.

[Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

[Tos18] Valentino Tosatti, *Kawa lecture notes on the kähler–ricci flow*, Annales de la Faculté des sciences de Toulouse : Mathématiques **27** (2018), no. 2, 285–376.

Currents and Lelong Numbers

Kähler MMP Yi Li

Contents

1 Potentials and positive currents 1

2 Compactness and pseudoeffective classes 3

3 Envelopes and minimal singularities 4

4 Lelong numbers and divisorial singularities 5

5 The hand-off to the MMP and research directions 6

Big picture

The transcendental MMP replaces effective divisors by positive currents and orders of vanishing by Lelong numbers. The passage from individual currents to cohomology classes depends on two existence arguments: weak compactness at fixed mass and the envelope construction of a least singular potential. We prove these arguments, explain exactly what the Lelong number measures, and identify the additional analytic theorems used in later chapters.

1 Potentials and positive currents

Throughout the main arguments X is a connected compact Kähler manifold of dimension n, ω is a fixed Kähler form, and θ is a smooth closed real $(1,1)$-form. Statements on singular spaces require the additional conventions discussed at the end. We use $d^c = \frac{i}{2\pi}(\bar\partial - \partial)$, so $dd^c = \frac{i}{\pi}\partial\bar\partial$.

Definition 1.1. A plurisubharmonic function is an upper semicontinuous function with values in $[-\infty, \infty)$, not identically $-\infty$, whose restriction to each holomorphic disc is subharmonic or identically $-\infty$. A quasi-plurisubharmonic function is locally a sum of a psh function and a smooth function. We write $\mathrm{PSH}(X, \theta)$ for the quasi-psh functions u satisfying $\theta + dd^c u \geq 0$ in the sense of currents.

A real $(1,1)$-current is a differential form with distribution coefficients. It is positive if it acts nonnegatively on smooth compactly supported strongly positive $(n-1,n-1)$-forms. Positivity implies that its coefficients are measures. Closedness means $dT = 0$ distributionally.

Proposition 1.2. *If u is psh, $dd^c u$ is a closed positive $(1,1)$-current. Conversely, a locally integrable function whose distributional dd^c is positive has a unique psh representative modulo equality almost everywhere. A closed positive $(1,1)$-current has psh potentials locally.*

Proof. On a relatively compact coordinate ball, convolve u with a nonnegative radial approximate identity. The resulting smooth functions are psh on smaller balls, since their restrictions to complex lines satisfy the submean inequality. Their complex Hessians are therefore positive. Convergence in L^1_{loc} gives positivity of $dd^c u$ by testing against positive forms; closedness follows from $d\,dd^c = 0$. Conversely, convolution of a distribution with positive dd^c has positive complex Hessian. The subharmonic representative theorem, applied locally and compatibly on overlaps, gives the psh representative. Uniqueness follows because a subharmonic function is recovered from its almost-everywhere values by limits of its ball averages. Finally the distributional local $\partial\bar\partial$-lemma gives a real distribution v with $T = dd^c v$ on a smaller ball. Since its Laplacian is a positive measure, the local potential theorem for the Laplacian represents v by an L^1_{loc} function up to a smooth harmonic function. The preceding argument makes its canonical representative psh. ∎

Remark 1.3. The representative clause cannot be dropped: changing an L^1 function at one point leaves its current unchanged but may destroy the submean inequality. Potentials in these notes always mean their psh or quasi-psh representatives.

Proposition 1.4. *Every positive current in the Bott–Chern class $\{\theta\}$ is $\theta + dd^c u$ for some $u \in \mathrm{PSH}(X,\theta)$. This potential is unique up to an additive constant.*

Proof. The Bott–Chern class identity, in the current resolution of Bott–Chern cohomology, gives a global real distribution u with $T - \theta = dd^c u$. Locally write $\theta = dd^c \rho$ with ρ smooth. By Proposition 1.2, $u + \rho$ has a psh representative, giving a global quasi-psh representative of u. If two such potentials differ by v, then $dd^c v = 0$. Elliptic regularity makes v smooth, and its Laplacian for a Kähler metric vanishes. Integration by parts gives $dv = 0$. Connectedness makes v constant. ∎

Example 1.5. If $f_1, \dots, f_r$ are holomorphic and not all identically zero, then $\log(\sum |f_i|^2)^{1/2}$ is psh. Away from their common zero set this follows by differentiating and applying Cauchy–Schwarz; across that set it follows by decreasing the smooth psh

functions $\frac{1}{2}\log(\sum |f_i|^2 + \varepsilon)$. For a nonzero holomorphic function f, the Poincaré–Lelong formula in our normalization is $dd^c \log|f| = [\mathrm{div}(f)]$. The one-variable identity follows from Green's formula around the origin; at a smooth point of the divisor it applies in a transverse coordinate. The general identity is the Poincaré–Lelong theorem [Dem12], including multiplicities along all components.

2 Compactness and pseudoeffective classes

Theorem 2.1 (Weak compactness at bounded mass)**.** *For each $M \geq 0$, the closed positive $(1,1)$-currents T on X with $\int_X T \wedge \omega^{n-1} \leq M$ form a sequentially compact set in the weak topology of currents.*

Proof. Choose finitely many coordinate charts and a subordinate partition of unity. On a chart, a positive $(1,1)$-current is a positive Hermitian matrix of measures. Its diagonal measures are nonnegative, and positivity of the 2×2 principal submatrices bounds the total variations of its off-diagonal coefficients by a fixed multiple of the trace measure. The chart metric and ω are uniformly comparable on the compact support of each partition function. Thus, for every smooth test form η,

$$|T(\eta)| \leq CM\|\eta\|_{C^0},$$

with C depending on the fixed finite atlas and metric, not on T. Choose a countable set of smooth test forms dense in the C^0 norm. Boundedness and diagonal extraction produce a subsequence converging on this set. The displayed estimate extends convergence to every smooth test form and defines an order-zero current T_∞. For every strongly positive test form η, the limit of $T_j(\eta) \geq 0$ is nonnegative. For every test form ξ of the appropriate degree, $T_\infty(d\xi) = \lim T_j(d\xi) = 0$. Thus the limit is positive and closed. Testing against ω^{n-1} gives its mass bound. ∎

Corollary 2.2. *The positive currents in a fixed Bott–Chern class form a weakly compact set. The pseudoeffective cone is closed.*

Proof. Stokes' theorem gives $\int_X T \wedge \omega^{n-1} = \{T\} \cdot \{\omega\}^{n-1}$. For a convergent sequence of classes these masses are bounded, so Theorem 2.1 gives a positive closed weak limit. Pairing with closed smooth forms identifies its de Rham class with the limiting class. The $\partial\bar\partial$-lemma on the compact Kähler manifold identifies real Bott–Chern $(1,1)$-classes with their de Rham images; hence the limit has exactly the required Bott–Chern class. ∎

Proposition 2.3. *Every nef class is pseudoeffective.*

Proof. Let α be nef. For every j there is a smooth closed representative $\theta_j \in \alpha$ with $\theta_j \geq -j^{-1}\omega$. Then $T_j = \theta_j + j^{-1}\omega$ is positive and closed, and its mass converges to $\alpha \cdot \{\omega\}^{n-1}$. By Theorem 2.1, a subsequence converges to a positive current in α, as in the proof of Corollary 2.2. ∎

3 Envelopes and minimal singularities

For potentials u, v, say that u is less singular than v if $v \leq u + C$ for some constant C.

Lemma 3.1 (The upper-envelope principle)**.** *If a nonempty family of θ-psh functions is locally uniformly bounded above, the upper semicontinuous regularization of its supremum is θ-psh.*

Proof. Locally add a smooth potential ρ of θ. The assertion becomes the psh upper-envelope theorem. We recall the reduction needed for an arbitrary, possibly uncountable, family. Choquet's lemma gives a countable subfamily whose upper-regularized supremum equals that of the original family. Finite maxima of its first j members form an increasing sequence of psh functions, locally uniformly bounded above. The increasing-limit theorem for psh functions says that the upper regularization of its limit is psh. Both Choquet's lemma and this local psh convergence theorem are standard potential-theoretic inputs, as in [Dem12]. A member of the original family supplies a psh lower bound, so the result is not identically $-\infty$. Subtracting ρ gives the assertion. ∎

Theorem 3.2 (Existence of a least singular potential)**.** *Suppose $\{\theta\}$ is pseudoeffective. Then*

$$V_\theta = \big(\sup\{u \in \mathrm{PSH}(X,\theta) : u \leq 0\}\big)^*$$

belongs to $\mathrm{PSH}(X,\theta)$ and is less singular than every other θ-psh potential. The corresponding current $T_{\min} = \theta + dd^c V_\theta$ has minimal singularities.

Proof. A positive current in $\{\theta\}$ has a global quasi-psh potential by Proposition 1.4. Upper semicontinuity and compactness give this potential a finite supremum, so subtracting its supremum makes the family in the formula nonempty. Its members are uniformly bounded above by zero. Apply Lemma 3.1; the resulting potential still satisfies $V_\theta \leq 0$. For any $v \in \mathrm{PSH}(X,\theta)$, the normalized potential $v - \sup_X v$ belongs to the family. Therefore $v \leq V_\theta + \sup_X v$, which is precisely the required comparison. If two currents have minimal singularities, comparison in both directions makes their potentials differ by a bounded function. This gives uniqueness of the singularity type, not uniqueness of the current. ∎

Remark 3.3. The current $T_{\min}$ exists even on the boundary of the pseudoeffective cone. Boucksom's minimal multiplicities nevertheless use the currents with minimal singularities in $\alpha + \varepsilon\{\omega\}$ and a limit as $\varepsilon \downarrow 0$. The reason for this perturbation is the

desired numerical invariant and its behavior at the boundary, not nonexistence of a least singular positive current in α.

4 Lelong numbers and divisorial singularities

For $n \geq 2$ put $\beta = dd^c|z|^2$ in local coordinates centred at x. For a positive closed $(1,1)$-current the normalized mass ratio is

$$\nu(T, x, r) = \frac{1}{2^{n-1} r^{2n-2}} \int_{|z|<r} T \wedge \beta^{n-1}.$$

For $n = 1$ the Lelong number is the atomic mass at x. Lelong's monotonicity theorem says that the displayed ratio is increasing in r, and its limit as $r \downarrow 0$ is independent of coordinates. Equivalently, if $T = dd^c u$ locally, then

$$\nu(T, x) = \sup\{c \geq 0 : u(z) \leq c \log|z - x| + O(1)\}.$$

We use the monotonicity and potential-comparison theorems of Lelong as analytic inputs [Dem12]; a proof of their equivalence requires the Lelong–Jensen formula.

Proposition 4.1. *If u is less singular than v, then $\nu(dd^c u, x) \leq \nu(dd^c v, x)$. Bounded changes of potential preserve Lelong numbers. Smooth forms have zero Lelong number.*

Proof. Write $v \leq u + C$. If $u \leq c \log|z - x| + O(1)$, the same upper bound holds for v, so the potential characterization gives the inequality. For bounded differences apply it twice. For a smooth $(1,1)$-form the mass in a ball is $O(r^{2n})$; division by r^{2n-2} tends to zero. In dimension one a smooth measure has no atom. ∎

Example 4.2. For $D = \sum a_i D_i$, with $a_i \geq 0$, the generic Lelong number of $[D]$ along D_i is a_i. At a smooth point belonging to only D_i, choose a coordinate with $D_i = (z_1 = 0)$ and apply $[D] = dd^c(a_i \log|z_1|)$ + a smooth local contribution. At singular points the formula is $\nu([D], x) = \sum a_i \operatorname{mult}_x D_i$, the multiplicity form of the Poincaré–Lelong theorem.

Theorem 4.3 (Siu's decomposition; cited input)**.** *A closed positive $(1,1)$-current admits a unique decomposition $T = \sum_D \nu(T, D)[D] + R$, where the sum ranges over prime divisors with positive generic Lelong number and R is positive, closed, and has zero generic Lelong number on every prime divisor.*

This deep theorem is the current-level input for divisorial Zariski decomposition. Its full proof, including analyticity of positive Lelong level sets and convergence of the divisorial sum, is a remaining proof expansion in this base draft; it is not supplied by the elementary comparison proposition above.

5 The hand-off to the MMP and research directions

The cone and positivity note uses Theorems 2.1 and 3.2 to pass from representatives to classes. The divisorial Zariski-decomposition note uses generic Lelong numbers and Siu's theorem to isolate divisorial obstructions. Contraction arguments also require regularization by currents with analytic singularities and the comparison of the null and non-Kähler loci. These are additional theorems, not formal consequences of weak compactness. In particular a maximum of logarithmic potentials need not have analytic singularities in the strict ideal-theoretic sense; the approximation step must be stated in the version actually used.

Question 5.1. *For which boundary classes α does the generic Lelong number of $T_{\min} \in \alpha$ equal the limiting minimal multiplicity obtained from $\alpha + \varepsilon\{\omega\}$? The first task is to determine which steps in the big-class argument use openness of the big cone.*

Question 5.2. *Which potential-theoretic constructions descend through a resolution of a normal compact Kähler space? One must specify local potentials and the cohomology theory, and separately check the dependence of Lelong numbers on an embedding at singular points.*

Question 5.3. *What additional control on multiplier ideals is needed when regularized currents enter an abundance argument? Convergence of Lelong numbers alone does not identify multiplier ideals; an argument using extension of sections must state the precise equisingularity input it needs.*

These are questions motivating the series. Their formulation here does not certify novelty or claim a current literature classification.

References

[Dem12] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, Higher Education Press, Beijing, 2012.

Positivities and Cones in the Kähler MMP

Kähler MMP Yi Li

The aim of this note is to give a brief introduction to positivities and cones in the Kähler minimal model program. On a general compact Kähler variety there are no divisors to work with, so that the positivity notion has to be reformulated on *cohomology classes and the curvature current of singular metrics*. This note is organized as follows: Section 1 introduce the working space, Bott–Chern cohomology $H^{1,1}_{\mathrm{BC}}(X)$, and the numerical spaces $N^1(X)$, $N_1(X)$. We then introduce several positivity notions include: Kählerness, bignness, pseudo-effectiveness, nefness and semi-ampleness. We end this note by introducing Mori cone and generalized Mori cone with some dualities of these cones.

Contents

1 Setting: Bott–Chern cohomology and the numerical spaces **2**

1.1 Bott–Chern cohomology 2

1.2 Comparison with $H^2(X,\mathbb{R})$: where rational singularities enter 3

1.3 Functoriality of Bott–Chern classes 4

1.4 The numerical spaces $N^1(X)$ and $N_1(X)$ 5

1.5 The dictionary 6

2 Positivities: Kähler, big, pseudoeffective, nef, semi-ample **6**

2.1 Kähler classes 7

2.2 Kähler currents and big classes 11

2.3 Pseudoeffective classes 14

2.4 Nefness 16

2.5 Semi-positivity and semi-ampleness 19

3 Mori cone, generalized Mori cone, and duality of cones **21**

3.1 The two Mori cones 21

3.2 Duality of cones 22

4 Appendix: Kählerness criterion **23**

4.1 The Demailly–Păun Nakai–Moishezon criterion 23

4.2 The Kähler cone is a connected component of the positive cone 23

4.3 The Kleiman-type criterion 24

4.4 Fujiki, Moishezon, projective 24

The major references are [DP04], [Bou04], [HP16], [DH24], [DHP24], [DHY23], [BDPP13] and [Dem12, Ch. 18–19].

1 Setting: Bott–Chern cohomology and the numerical spaces

Convention 1.1. Throughout, X is a normal compact complex variety in Fujiki's class $\mathscr{C}$ unless stated otherwise, and $n = \dim X$. All sheaves below are sheaves of *real-valued* objects. We write $\mathcal{A}^0_X$ for smooth functions $U \to \mathbb{R}$ (on a singular X: restrictions of smooth functions under a local embedding $U \hookrightarrow \Omega \subset \mathbb{C}^N$), $\mathcal{D}_X$ for degree-0 currents, and $\mathcal{H}_X$ for the sheaf of **pluriharmonic** functions, locally $u = \operatorname{Im} f$ with f holomorphic. The operator d^c is normalised so that $dd^c = \frac{i}{\pi}\partial\bar\partial$; the constant plays no role below.

1.1 Bott–Chern cohomology

Definition 1.2 (Höring–Peternell, [HP16, Definition 3.1])**.** Let X be an irreducible reduced complex space. The **Bott–Chern cohomology** of X in bidegree $(1,1)$ is

$$H^{1,1}_{\mathrm{BC}}(X) := H^1(X, \mathcal{H}_X).$$

On a complex manifold this agrees with the two familiar descriptions,

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) = \left(\frac{\ker\partial \cap \ker\bar\partial}{\operatorname{im}\partial\bar\partial}\right)^{p,q} = \left(\frac{\ker d}{\operatorname{im}\partial\bar\partial}\right)^{p,q},$$

and with the same quotient formed using closed $(1,1)$-*currents*: every closed $(1,1)$-current is Bott–Chern cohomologous to a smooth form. It is the sheaf-theoretic description of Definition 1.2 that survives on a singular space, which is why it is taken as the definition here.

Definition 1.3. A **$(1,1)$-form (resp. current) with local potentials** on X is a global section of $\mathcal{A}^0_X/\mathcal{H}_X$ (resp. $\mathcal{D}_X/\mathcal{H}_X$).

Unwinding the sheaf quotient, such a section is exactly the data of an open cover $\{U_i\}$, functions $\varphi_i \in \mathcal{A}^0(U_i)$ — the **local potentials** — and the compatibility $\varphi_i - \varphi_j \in \mathcal{H}_X(U_i \cap U_j)$

on overlaps. Since pluriharmonic functions are annihilated by dd^c, the compatibility gives $dd^c\varphi_i = dd^c\varphi_j$ on overlaps, so the local forms glue to one global closed real $(1,1)$-form θ with $\theta|_{U_i} = dd^c\varphi_i$. No global potential is asserted to exist.

Proposition 1.4. *Every Bott–Chern class is represented by a $(1,1)$-form (equivalently, a current) with local potentials.*

Proof. The sheaf $\mathcal{A}^0_X$ is fine, hence acyclic, so $H^1(X,\mathcal{A}^0_X) = 0$. The connecting map of

$$0 \to \mathcal{H}_X \to \mathcal{A}^0_X \to \mathcal{A}^0_X/\mathcal{H}_X \to 0$$

therefore gives a surjection $H^0(X,\mathcal{A}^0_X/\mathcal{H}_X) \twoheadrightarrow H^1(X,\mathcal{H}_X) = H^{1,1}_{\mathrm{BC}}(X)$. ∎

Remark 1.5. The local potential, not the class, is what the theory actually reads. Positivity, Lelong numbers and the whole singularity theory of Sections 2.2 and 2.3 are conditions on φ_i. On a singular X there is no other way to define any of them.

1.2 Comparison with $H^2(X,\mathbb{R})$: where rational singularities enter

Proposition 1.6. *Let X be a normal compact complex space in class $\mathcal{C}$ with rational singularities. Then the natural map*

$$H^{1,1}_{\mathrm{BC}}(X) \hookrightarrow H^2(X,\mathbb{R})$$

is injective. In particular a Bott–Chern class may be treated as a topological class, and $H^{1,1}_{\mathrm{BC}}(X)$ is finite-dimensional.

Proof. The sheaf $\mathcal{H}_X$ sits in the fundamental exact sequence

$$0 \to \mathbb{R} \to \mathcal{O}_X \xrightarrow{2\,\mathrm{Im}} \mathcal{H}_X \to 0,$$

(a holomorphic function with vanishing imaginary part is constant, by the Cauchy–Riemann equations) whose long exact sequence reads

$$H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X) \to H^{1,1}_{\mathrm{BC}}(X) \to H^2(X,\mathbb{R}) \to H^2(X,\mathcal{O}_X).$$

So $H^{1,1}_{\mathrm{BC}}(X) \to H^2(X,\mathbb{R})$ is injective precisely when $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X)$ is surjective. We check the latter on a resolution $\pi\colon \hat{X} \to X$, using two comparison isomorphisms. In each, normality gives injectivity for free and rationality is what gives surjectivity.
Coherent side. Normality gives $\pi_*\mathcal{O}_{\hat{X}} = \mathcal{O}_X$, and rational singularities are by definition $R^q\pi_*\mathcal{O}_{\hat{X}} = 0$ for $q > 0$. So the Leray spectral sequence degenerates and $H^k(X,\mathcal{O}_X) \cong H^k(\hat{X},\mathcal{O}_{\hat{X}})$ for all k.
Topological side. Here π has connected fibres by Zariski's main theorem, so $\pi_*\mathbb{R}_{\hat{X}} = \mathbb{R}_X$,

and the five-term exact sequence of Leray reads

$$0 \to H^1(X,\mathbb{R}) \to H^1(\hat{X},\mathbb{R}) \to H^0(X, R^1\pi_*\mathbb{R}) \to H^2(X,\mathbb{R}).$$

Injectivity is automatic on any normal X; what is needed is $R^1\pi_*\mathbb{R} = 0$, and this is exactly where rationality is spent.
In the commutative square formed by these two isomorphisms and the two maps $H^1(-,\mathbb{R}) \to H^1(-,\mathcal{O})$, surjectivity on X is equivalent to surjectivity on $\hat{X}$. There, degree-one Hodge decomposition $H^1(\hat{X},\mathbb{C}) = H^{1,0} \oplus H^{0,1}$ makes $H^1(\hat{X},\mathbb{R}) \to H^1(\hat{X},\mathcal{O}_{\hat{X}})$ an isomorphism of *real* vector spaces. ∎

Remark 1.7. This is the first of many places where rational singularities does real work, and it is what one is actually buying: a well-defined intersection theory on $H^{1,1}_{\mathrm{BC}}(X)$, inherited from $H^2(X,\mathbb{R})$.

1.3 Functoriality of Bott–Chern classes

A current cannot in general be pulled back. A $(1,1)$-current *with local potentials* can: one pulls back the potentials, and the pulled-back object again has local potentials. So f^* is defined on $H^{1,1}_{\mathrm{BC}}$. What is not automatic is injectivity.

Theorem 1.8 ([HP16, Lemma 3.3]; [DHP24, Lemma 8.7])**.** *Let $f : X \to Y$ be a proper morphism with connected fibres between normal compact complex spaces with rational singularities. Assume either*

1. *$X, Y \in \mathcal{C}$ and f is bimeromorphic, or*
2. *there is an effective $\mathbb{Q}$-divisor $B \geq 0$ with (X,B) klt, $-(K_X + B)$ f-nef and f-big, and f projective.*

Then

$$f^*\colon H^{1,1}_{\mathrm{BC}}(Y) \to H^{1,1}_{\mathrm{BC}}(X), \qquad f^*\colon H^2(Y,\mathbb{R}) \to H^2(X,\mathbb{R})$$

are both injective, and

$$\mathrm{Im}(f^*) = \{\alpha \in H^{1,1}_{\mathrm{BC}}(X) \;:\; \alpha \cdot C = 0 \text{ for every curve } C \subset X \text{ contracted by } f\}.$$

Moreover such an α equals $f^*[\omega_Y]$ for a smooth *real closed $(1,1)$-form ω_Y with local potentials on Y.*

Proof of injectivity. Apply the five-term exact sequence of the Leray spectral sequence to $\mathcal{F} = \mathcal{H}_X$, using $f_*\mathcal{H}_X = \mathcal{H}_Y$ (the direct image of the sheaf of pluriharmonic functions under a proper bimeromorphic morphism is again the sheaf of pluriharmonic functions):

$$0 \to H^1(Y,\mathcal{H}_Y) \to H^1(X,\mathcal{H}_X) \to H^0(Y, R^1 f_*\mathcal{H}_X) \to H^2(Y,\mathcal{H}_Y) \to H^2(X,\mathcal{H}_X).$$

Exactness at the second place is injectivity of $f^*\colon H^{1,1}_{\mathrm{BC}}(Y) \to H^{1,1}_{\mathrm{BC}}(X)$. For the topological

statement, the hypotheses give $R^q f_*\mathcal{O}_X = 0$ for $q \geq 1$ and $R^1 f_*\mathbb{R} = 0$, so Leray for $\mathbb{R}$ gives $0 \to H^2(Y,\mathbb{R}) \to H^2(X,\mathbb{R}) \to H^0(Y, R^2 f_*\mathbb{R})$. ∎

Idea for the description of $\mathrm{Im}(f^*)$. Pushing the sequence $0 \to \mathbb{R} \to \mathcal{O}_X \to \mathcal{H}_X \to 0$ forward and using $R^{\geq 1} f_*\mathcal{O}_X = 0$, $R^1 f_*\mathbb{R} = 0$ gives an isomorphism $R^1 f_*\mathcal{H}_X \cong R^2 f_*\mathbb{R}$. This makes the two Leray sequences above into a commutative ladder with vertical injections from Proposition 1.6 and an isomorphism on the right. A class dies in $H^0(Y, R^2 f_*\mathbb{R})$ exactly when it is orthogonal to every 2-cycle contained in a fibre; since H_2 of a fibre is generated by algebraic cycles, that is exactly the stated numerical condition. Full details are in [HP16, DHP24].

Remark 1.9. This theorem is the transcendental half of the projective implication "$L \cdot R = 0 \Rightarrow L = f^*L_Z$". The descent statement is available cheaply here; what is hard is the *second* step, turning the descended class into a Kähler class downstairs. That is the content of the base point free theorem (Theorem 2.52).

Push-forward is more delicate: currents always push forward, but on a singular space the push-forward of a current with local potentials need not have local potentials.

1.4 The numerical spaces $N^1(X)$ and $N_1(X)$

Definition 1.10 ([HP16, Definitions 3.6 and 3.8])**.** Let X be a normal compact complex space in class $\mathcal{C}$.

1. $N^1(X) := H^{1,1}_{\mathrm{BC}}(X)$.
2. $N_1(X)$ is the vector space of real closed currents of bidimension $(1,1)$ modulo the relation $T_1 \equiv T_2 \iff T_1(\eta) = T_2(\eta)$ for every real closed $(1,1)$-form η with local potentials.

Remark 1.11. Note the deliberate choice in (1): in the analytic setting one does *not* quotient $H^{1,1}_{\mathrm{BC}}$ further by numerical equivalence. The transcendental N^1 *is* the Bott–Chern group. The algebraic N^1 generated by numerical classes of irreducible divisors is in general far too small to carry the theory.

Proposition 1.12 ([HP16, Proposition 3.9])**.** *Let X be a normal compact complex space of dimension n with rational singularities, in class $\mathcal{C}$. Then*

$$\Phi\colon N^1(X) \longrightarrow N_1(X)^\vee, \qquad [\omega] \longmapsto \big([T] \mapsto T(\omega)\big)$$

is an isomorphism. In particular $N_1(X)$ is finite-dimensional, and every linear functional $\lambda\colon N_1(X) \to \mathbb{R}$ has the form $\lambda([T]) = T(\omega)$ for a real closed $(1,1)$-form ω with local potentials.

Proof. Injectivity of Φ is the definition of $N^1(X)$: if $T(\omega) = 0$ for every closed current of bidimension $(1,1)$ then $[\omega] = 0$. For surjectivity, $N^1(X)$ is finite-dimensional by Proposition 1.6, so it suffices to see $\dim N_1(X) \le \dim N^1(X)$; and the map $\psi\colon N_1(X) \to N^1(X)^*$, $[T] \mapsto ([\omega] \mapsto T(\omega))$, is injective by the definition of $N_1(X)$. For the last statement, pass to a resolution $\pi\colon \hat{X} \to X$: Theorem 1.8 says $N^1(X) \to N^1(\hat{X})$ is injective, and dualising turns this into surjectivity of $\pi_*\colon N_1(\hat{X}) \to N_1(X)$. ∎

Curves land in $N_1(X)$ because the current of integration over a curve is closed and positive: for an irreducible curve $C \subset X$ we write $[C]\colon H^{1,1}_{\mathrm{BC}}(X) \ni \eta \mapsto \int_C \eta$.

> **Warning.** Every notion below — nef in particular — is defined only for compact X. Without properness the definitions say nothing.

1.5 The dictionary

The whole subject is organised by a single substitution.

projective	**Kähler**
ample divisor	Kähler class
effective divisor	closed positive current
big divisor (sections grow)	class containing a Kähler current
pseudoeffective divisor	class containing a positive current
$\overline{\mathrm{NE}}(X)$, curves	$\overline{\mathrm{NA}}(X)$, positive currents of bidimension $(1,1)$
linear system, base locus	Lelong numbers, non-Kähler locus

2 Positivities: Kähler, big, pseudoeffective, nef, semi-ample

The four tiers of positivity, treated as objects: how each is defined, what each is stable under, and how they sit inside one another. The organising principle is stated once here and then used repeatedly.

> *Kähler and big are* open *conditions; nef and pseudoeffective are* closed *ones.*

Open conditions survive perturbation and are what one carries along inside a construction; closed conditions survive passage to the limit and are what one concludes at the end of one. Each pair is related by taking interiors and closures (Corollary 2.6, Proposition 2.25), and each tier absorbs the weaker one on its own side (Lemma 2.5, Proposition 2.25). The fifth tier, semi-ampleness, is different in kind: it is what turns a class into a morphism, and it is treated last.

2.1 Kähler classes

Definition 2.1 ([HP16, Definition 2.2])**.** An analytic variety X is **Kähler** if there is a positive closed real $(1,1)$-form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ such that for every $x \in X_{\mathrm{sing}}$ there are an open neighbourhood $x \in U \subseteq X$, a closed embedding $\iota_U : U \hookrightarrow V \subseteq \mathbb{C}^N$, and a smooth strictly plurisubharmonic $f : V \to \mathbb{R}$ with $\omega|_{U \cap X_{\mathrm{sm}}} = (i\partial\bar{\partial} f)|_{U \cap X_{\mathrm{sm}}}$. The **Kähler cone** $\mathcal{K}_X \subset H^{1,1}_{\mathrm{BC}}(X)$ is the cone generated by the classes of Kähler forms.

Intuition. Two features of the definition, rather than the definition itself, are what get used. First, a Kähler form is locally dd^c of a function, so it is a form with local potentials in the sense of Definition 1.3 and therefore has a Bott–Chern class: this is why the theory can be run on $H^{1,1}_{\mathrm{BC}}$ and not on some larger group. Second, the potential is *strictly* plurisubharmonic. Every weaker tier below is obtained by degrading exactly this word — psh instead of strictly psh, or a lower bound $-\epsilon\omega$ instead of ≥ 0.

Observation 2.2. A Kähler class is nonzero.

Proof. A global potential φ with $\omega = dd^c\varphi$ would be a psh function on a compact space, hence constant, forcing $\omega = 0$. ∎

2.1.1 The Kähler cone is open, convex, and contains no line

Proposition 2.3. *Let X be a compact Kähler manifold. Then $\mathcal{K}_X \subset H^{1,1}_{\mathrm{BC}}(X)$ is a non-empty open convex cone containing no straight line.*

Proof. Convexity and the cone property are immediate: a sum of two Kähler forms is Kähler, and a positive multiple of a Kähler form is Kähler. Salience follows: if α and $-\alpha$ were both Kähler then $0 = \alpha + (-\alpha)$ would be Kähler, contradicting Observation 2.2. Openness is Lemma 2.5 below, which gives $\alpha + \omega \in \mathcal{K}_X$ for every $\omega \in \mathcal{K}_X$ and every α in a neighbourhood of 0 (such an α is a difference of nef classes). ∎

Remark 2.4. Salience is what makes extremal-ray arguments possible at all: a salient closed convex cone is the convex hull of its extremal rays. The same property holds for $\mathrm{Nef}(X)$, $\mathcal{E}_X$ and $\overline{\mathrm{NA}}(X)$.

2.1.2 Kähler + nef = Kähler

This one lemma carries a large part of the formal structure of the subject. Its statement uses the definition of nefness from Definition 2.39; nothing else in Section 2.4 is needed.

Lemma 2.5. *If ω is a Kähler class and α is nef, then $\alpha + \omega$ is Kähler. The same holds if α is merely closed semi-positive with local potentials.*

Proof. By Definition 2.39 there is, for each $\epsilon > 0$, a smooth f_ϵ with $\alpha + i\partial\bar\partial f_\epsilon \geq -\epsilon\omega$. Taking $\epsilon < 1$,

$$\alpha + \omega + i\partial\bar\partial f_\epsilon \;\geq\; (1-\epsilon)\,\omega \;>\; 0,$$

which is a Kähler form in the class $\alpha + \omega$. If α is semi-positive, take $f_\epsilon = 0$. ∎

Corollary 2.6. $\mathrm{Nef}(X) = \overline{\mathcal{K}_X}$ *and* $\mathcal{K}_X = \operatorname{int}\mathrm{Nef}(X)$.

Remark 2.7. Every "perturb and stay positive" step in the Kähler MMP is Lemma 2.5. It is why a nef class is a limit of Kähler classes, why one may add a small Kähler class to restore strict positivity before applying a criterion, and why perturbing by a very general Kähler class to isolate a single extremal ray is legitimate. The semi-positive case is what is used when α is a pull-back $f^*\omega_Y$ (Section 2.5).

2.1.3 Strict positivity on subvarieties

Lemma 2.8 (Barth–Hulek–Peters–Van de Ven, Lemma 13.1)**.** *Let X be a compact Kähler manifold, ω a Kähler class and $V \subset X$ an irreducible analytic subvariety with $p = \dim V > 0$. Then*

$$\int_V \omega^p \;>\; 0.$$

Proof. The restriction $\omega|_{V_{\mathrm{reg}}}$ is a Kähler form on V_{reg}, so $\omega^p|_{V_{\mathrm{reg}}}$ is a positive volume form and its integral is positive. ∎

Corollary 2.9. *A Kähler class is nef and big, and is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$. In particular $\mathcal{K}_X \subseteq \mathcal{B}_X \cap \mathrm{Nef}(X)$.*

Proof. Nefness is immediate from Definition 2.39 with $f_\epsilon = 0$. Taking $V = X$ in Lemma 2.8 gives $\int_X \omega^n > 0$, hence $v(\omega) > 0$ and bigness by Theorem 2.23; alternatively ω is itself a Kähler current. Positivity on $\overline{\mathrm{NA}}(X)$ is Lemma 2.8 applied to the generators $\int_V (-) \wedge \omega^{d-1}$ of that cone (Section 3). ∎

Remark 2.10. The conditions in Lemma 2.8 are *necessary*. How far they are from sufficient is the entire content of the Demailly–Păun theory (Section 4), and the answer is: sufficient, up to passing to the correct connected component.

2.1.4 Pull-back, push-forward and restriction of Kähler classes

The pattern is that Kählerness restricts well, pulls back badly, and pushes forward only as a current.

Lemma 2.11 (Restriction). *If $\iota : V \hookrightarrow X$ is a subvariety and ω is Kähler on X, then $\iota^*\omega$ is Kähler on V.*

Proof. Positivity and closedness of a form are preserved by pull-back along an immersion. For the local potentials: around a point of V, compose the local embedding $U \hookrightarrow \Omega \subset \mathbb{C}^N$ of Definition 2.1 with $V \cap U \hookrightarrow U$, so the same strictly psh f on Ω serves as a local potential for $\iota^*\omega$. A strictly psh function restricted to a complex submanifold of Ω is again strictly psh, since the complex Hessian restricts to the complex Hessian. ∎

Fact 2.12 (Pull-back). For $f : X \to Y$ bimeromorphic or a fibre space, $f^*\omega_Y$ is only *semi-positive*: the complex Hessian of $f^*\varphi$ degenerates in the directions tangent to the fibres of f, exactly where df drops rank. So $f^*\omega_Y$ is a closed semi-positive form with local potentials, and it is Kähler only if f is finite and unramified.

Lemma 2.13 ([DH24]). *Let $f : X \to Y$ be a birational morphism, α_Y a Kähler form on Y and α_X a relative Kähler form. Then $s\, f^*\alpha_Y + \alpha_X$ is Kähler for $s \gg 0$.*

Remark 2.14. Lemma 2.13 is the transcendental replacement for "ample + relatively ample = ample", and it is the workhorse for propagating the Kähler condition down one step of the MMP. In the $\mathbb{Q}$-factorial case one takes $\alpha_X = -E$ for a relatively ample exceptional divisor E and concludes that $s f^*\alpha_Y - E$ is Kähler.

Fact 2.15 (Blow-ups). If $\pi : \hat{X} \to X$ is the blow-up of a compact Kähler manifold along a smooth centre with exceptional divisor E, then $\pi^*\omega - \epsilon E$ is Kähler for $0 < \epsilon \ll 1$. In particular the blow-up of a Kähler manifold is Kähler.

The same "subtract a small exceptional class" device reappears in the proof of the Kleiman-type criterion (Theorem 4.4).

Fact 2.16 (Push-forward). For π bimeromorphic, $\pi_*\omega$ is in general only a Kähler current (Definition 2.20). This is precisely the mechanism that manufactures Kähler currents in Section 4. Under a rational-singularity hypothesis one can sometimes recover an honest smooth form, by taking a regularised maximum of the local potential with a smooth strictly psh function [HP16, Section 3].

2.1.5 The projectivity criterion

Openness of the Kähler cone (Proposition 2.3) has one consequence worth isolating, because it is the only bridge in these notes between the transcendental world and the projective one. Recall that

$$\mathrm{NS}(X)_{\mathbb{R}} := \left(H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Q})\right) \otimes_{\mathbb{Q}} \mathbb{R}$$

is the real Néron–Severi space, i.e. the classes of $\mathbb{R}$-divisors.

Theorem 2.17 (Projectivity criterion, $\mathbb{R}$-divisor form)**.** *Let X be a compact Kähler manifold. The following are equivalent.*

1. *X is projective.*
2. *$\mathcal{K}_X \cap H^2(X,\mathbb{Z}) \neq \emptyset$: some Kähler class is integral.*
3. *$\mathcal{K}_X \cap \mathrm{NS}(X)_{\mathbb{R}} \neq \emptyset$: some Kähler class is the class of an $\mathbb{R}$-divisor.*

Moreover, if X is projective then $\mathcal{K}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$ is exactly the ample cone of X.

Proof. (2)⇒(1) is the Kodaira embedding theorem, and (1)⇒(2) is the class of an ample divisor. (2)⇒(3) is trivial, since an integral class lies in $\mathrm{NS}(X)_{\mathbb{R}}$.
For (3)⇒(2), let $\alpha \in \mathcal{K}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$. By Proposition 2.3 the cone $\mathcal{K}_X$ is open in $H^{1,1}(X,\mathbb{R})$, so $\mathcal{K}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$ is a non-empty open subset of $\mathrm{NS}(X)_{\mathbb{R}}$. Rational points are dense in $\mathrm{NS}(X)_{\mathbb{R}}$ by construction, so this open set contains a class β with $\beta \in H^2(X,\mathbb{Q})$. Clearing denominators, $m\beta$ is an integral Kähler class for suitable $m \in \mathbb{Z}_{>0}$, since $\mathcal{K}_X$ is a cone.
For the last statement: an ample class is Kähler and lies in $\mathrm{NS}(X)_{\mathbb{R}}$; conversely a Kähler class in $\mathrm{NS}(X)_{\mathbb{R}}$ is a positive real combination of integral Kähler classes by the density argument just given, hence a positive combination of ample classes, hence ample. ∎

Remark 2.18. It is the *openness* of $\mathcal{K}_X$ that does the work: it converts a real condition into a rational one for free. This is the first of several places where "Kähler is an open condition" is not a technical convenience but the whole argument. It also explains the shape of the failure: a general compact Kähler manifold has $\mathrm{NS}(X)_{\mathbb{R}}$ too small — possibly zero — for the intersection to be non-empty, and no amount of positivity on X will fix that.

Example 2.19 (K3 surfaces)**.** On a K3 surface the Kähler cone is cut out inside the positive cone of the hyperbolic lattice $H^{1,1}(X,\mathbb{R})$ by the conditions $\omega \cdot C > 0$ over the (-2)-curves, and the ample cone is its intersection with $\mathrm{NS}(X)$, in accordance with Theorem 2.17. This is the cleanest place to see Proposition 2.3, Lemma 2.8 and Theorem 2.17 acting together.

2.2 Kähler currents and big classes

Definition 2.20 ([DH24, Definition 2.2])**.** Let X be a normal compact analytic variety. A closed positive $(1,1)$-current T on X with local potentials is a **Kähler current** if $T \geq \omega$ for some smooth Hermitian form ω on X. A class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is **big** if it contains a Kähler current.

Intuition. Bigness is defined *transcendentally*, through curvature: no global sections are involved, so the notion makes sense when there is no linear system at all. A big class is exactly a *degenerate Kähler class* — the Kähler form is allowed to acquire singularities, and the whole difficulty of the theory is controlling where.

2.2.1 Four descriptions of bigness

Theorem 2.21. *Let X be compact Kähler with a fixed Kähler form ω and $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$. The following are equivalent.*

1. *α contains a Kähler current.*
2. *$\alpha - \epsilon\{\omega\}$ is pseudoeffective for some $\epsilon > 0$.*
3. *α contains a Kähler current with analytic singularities.*
4. *$v(\alpha) > 0$.*

Proof. (1) $\Longleftrightarrow$ (2) is the definition read twice: a current $T \in \alpha$ with $T \geq \epsilon\omega$ is the same thing as a positive current $T - \epsilon\omega \in \alpha - \epsilon\{\omega\}$. (1)$\Rightarrow$(3) is Demailly regularization [Dem92], and (3)$\Rightarrow$(1) is trivial. (1) $\Longleftrightarrow$ (4) is Theorem 2.23. ∎

Remark 2.22. Implication (1)$\Rightarrow$(3) is the one that is actually used. A Kähler current with analytic singularities is a genuine Kähler metric on a Zariski open subset, so bigness becomes a statement one can do geometry with. This is the precise sense in which "big + nef ≃ singular Kähler metric".

Theorem 2.23 (Grauert–Riemenschneider criterion; [Bou02, Theorem 4.8])**.** *A pseudoeffective class α on a compact Kähler manifold is big if and only if $v(\alpha) > 0$.*

Idea of proof. One direction is immediate: if $T \geq \omega$ is a Kähler current in α then $T_{ac} \geq \omega$, because $T_{ac} - \omega = (T - \omega)_{ac}$ is the absolutely continuous part of a positive current; hence $\int_X T_{ac}^n \geq \int_X \omega^n > 0$. The converse is the substantial half: take $S_k \in \alpha$ with $\int_X S_{k,ac}^n \to v(\alpha)$, regularise to currents T_k with analytic singularities and $T_k \geq -\varepsilon_k\omega$,

resolve $\mu_k^\star T_k = \theta_k + E_k$, and apply the nef-and-big criterion to $\alpha_k := \{\theta_k + \varepsilon_k \mu^\star \omega\}$ to produce a Kähler current in $\alpha + \varepsilon_k\{\omega\}$. The whole point of the argument is that the resulting lower bound $\delta > 0$ can be taken uniform in k; this is where the estimates must be done by hand.

Remark 2.24. Theorem 2.23 is the transcendental replacement for "$h^0(mL)$ grows like m^n". It says the curvature-side and growth-side definitions of bigness still agree when there are no sections to count.

2.2.2 The big cone is open, and absorbs pseudoeffective classes

Proposition 2.25. *Let $\mathcal{B}_X$ be the big cone and $\mathcal{E}_X$ the pseudoeffective cone. Then*

1. $\mathcal{B}_X + \mathcal{E}_X \subseteq \mathcal{B}_X$;
2. $\mathcal{B}_X = \operatorname{int}(\mathcal{E}_X)$ *and* $\mathcal{E}_X = \overline{\mathcal{B}_X}$.

Proof. (1) If $T \geq \epsilon\omega$ is a Kähler current in α and $S \geq 0$ is a positive current in β, then $T + S \geq \epsilon\omega$ is a Kähler current in $\alpha + \beta$. (2) By Theorem 2.21(2), α is big iff $\alpha - \epsilon\{\omega\} \in \mathcal{E}_X$ for some $\epsilon > 0$, which says exactly that α is interior to $\mathcal{E}_X$; the second equality follows since $\mathcal{E}_X$ is closed (Proposition 2.32). ∎

Since $\mathcal{K}_X \subset \mathcal{B}_X$, part (1) contains "big + Kähler = big" and "big + nef = big" as special cases. This is the exact analogue of Lemma 2.5 one tier down, and is used the same way: to perturb without leaving the cone.

2.2.3 Pull-back and push-forward of big classes

Proposition 2.26. *Let $\pi\colon \hat{X} \to X$ be a proper bimeromorphic morphism of compact Kähler varieties and $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$. Then*

$$\alpha \text{ is big} \iff \pi^*\alpha \text{ is big},$$

and if $\hat{\alpha} \in H^{1,1}_{\mathrm{BC}}(\hat{X})$ is big then $\pi_*\hat{\alpha}$ is big.

Proof. ($\Rightarrow$) Let $T \in \alpha$ be a Kähler current, $T \geq \epsilon\omega_X$. Then $\pi^*T \in \pi^*\alpha$ is a closed positive current with local potentials (pull back the potentials) and $\pi^*T \geq \epsilon\,\pi^*\omega_X$. Now $\pi^*\{\omega_X\}$ is nef by Proposition 2.48 and big, since $\int_{\hat{X}} (\pi^*\omega_X)^n = \int_X \omega_X^n > 0$ and Theorem 2.23 applies. So $\pi^*\{\omega_X\}$ contains a Kähler current $S \geq \delta\omega_{\hat{X}}$, and $\pi^*T + \epsilon(S - \pi^*\omega_X)$ is a Kähler current in $\pi^*\alpha$.
($\Leftarrow$) Let $\hat{T} \in \pi^*\alpha$ satisfy $\hat{T} \geq \epsilon\,\pi^*\omega_X$. Push forward: π_* preserves positivity of currents, so

$\pi_*\hat{T} - \epsilon\,\pi_*\pi^*\omega_X \geq 0$. Since π is bimeromorphic, $\pi_*\pi^*\omega_X = \omega_X$, so $\pi_*\hat{T} \geq \epsilon\omega_X$ is a Kähler current in α. The same computation with $\pi^*\omega_X$ replaced by $\omega_{\hat{X}}$ and $\pi_*\omega_{\hat{X}}$ a Kähler current downstairs gives the statement for a general big $\hat{\alpha}$. ∎

Remark 2.27. This two-sided invariance is what makes bigness usable along an MMP: bigness of a class can be tested on any bimeromorphic model. It is the opposite of the behaviour of nefness (Proposition 2.48), which pulls back but does not push forward. A relative-to-absolute statement in the same family: a relatively big class plus a large multiple of the pull-back of a big class from the base is big — the bigness counterpart of Lemma 2.13.

2.2.4 Restriction of big and nef classes

Restriction behaves quite differently for the two notions, and the difference is worth stating plainly, because it is a common source of error.

Proposition 2.28. *Let $\iota\colon V \hookrightarrow X$ be an irreducible subvariety and $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ nef. Then $\alpha|_V$ is nef.*

Proof. Fix a Kähler form ω on X. By Proposition 2.40(1), $\alpha + \epsilon\{\omega\}$ is Kähler for every $\epsilon > 0$; by Lemma 2.11 its restriction $(\alpha + \epsilon\{\omega\})|_V$ is Kähler on V. That is, $\alpha|_V + \epsilon\{\omega|_V\}$ is Kähler for every $\epsilon > 0$, which by Proposition 2.40(1) applied on V says exactly that $\alpha|_V$ is nef. ∎

⚠ **Warning.** Bigness does not restrict. If α is big and $V \subset X$ is a subvariety, $\alpha|_V$ need not be big: a Kähler current $T \in \alpha$ may degenerate along V, and restricting it there destroys the strict lower bound $T \geq \epsilon\omega$. The correct statement quantifies over the locus where the current degenerates: if V is not contained in the locus where every Kähler current in α has positive Lelong numbers — the transcendental analogue of the augmented base locus — then $\alpha|_V$ is big. This is why, in practice, one restricts *nef and big* classes to *general* members of a family rather than to an arbitrary subvariety.

Lemma 2.29 ([DH25, Lemma 5.3])**.** *Let X be a normal compact Kähler variety and $f : X \to C$ a proper surjective morphism to a smooth projective curve. If $\omega \in H^{1,1}_{\mathrm{BC}}(X)$ is nef and big, then $\omega|_F$ is nef and big for a general fibre F of f.*

The nef half of Lemma 2.29 is Proposition 2.28 and holds for *every* fibre; it is the big half that needs the fibre to be general.

Fact 2.30. A nef and big class decomposes as a modified Kähler class plus an effective divisor.

2.3 Pseudoeffective classes

Definition 2.31. A class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is **pseudoeffective** if it contains a closed positive $(1,1)$-current. The pseudoeffective cone is denoted $\mathcal{E}_X$.

The two elementary comparisons: α is pseudoeffective iff $\alpha + \epsilon\omega$ is big for every $\epsilon > 0$ (Proposition 2.25), and nef $\Rightarrow$ pseudoeffective (Proposition 2.40).

2.3.1 The pseudoeffective cone is closed

Proposition 2.32. *$\mathcal{E}_X \subset H^{1,1}_{\mathrm{BC}}(X)$ is a closed convex cone containing no line.*

Proof. Convexity is immediate: a sum of positive currents is positive. For closedness, let $\alpha_k \to \alpha$ with $\alpha_k \in \mathcal{E}_X$, and choose positive currents $T_k \in \alpha_k$. The mass $\int_X T_k \wedge \omega^{n-1}$ is a cohomological quantity, computed from α_k alone, hence bounded along a convergent sequence of classes. Bounded sets of positive currents are weakly compact, so after passing to a subsequence $T_k \to T$ weakly, with $T \geq 0$ and T closed. Since the class map is continuous for the weak topology, $T \in \alpha$, so $\alpha \in \mathcal{E}_X$. Salience: if $\alpha, -\alpha \in \mathcal{E}_X$ then α is represented by a positive current T and $-\alpha$ by a positive current S, so $T + S$ is a positive current in the class 0; pairing with ω^{n-1} gives $T + S = 0$, hence $T = S = 0$ and $\alpha = 0$. ∎

Warning. $\mathcal{E}_X$ is closed, but it is *not* the closure of a cone of effective divisors — on a general Kähler variety there may be no effective divisors at all. It is defined by currents from the start. This is the same phenomenon that forced the introduction of $H^{1,1}_{\mathrm{BC}}$ in Section 1: on the divisor side there is nothing to take a closure of.

2.3.2 Siu decomposition

Theorem 2.33 (Siu). *Every closed positive $(1,1)$-current T decomposes as*

$$T = \sum_j \nu(T, D_j)\,[D_j] + R,$$

where the D_j are the prime divisors with $\nu(T, D_j) > 0$ (a countable, locally finite family) and R is a closed positive current all of whose divisorial Lelong numbers vanish.

Remark 2.34. Theorem 2.33 is the structural statement that makes the whole "modified" tier possible. Applied to a current with minimal singularities in α, it produces the divisorial Zariski decomposition $\alpha = P(\alpha) + N(\alpha)$ with $P(\alpha)$ modified nef and $N(\alpha)$ effective; the residual part R is what modified nefness is designed to see. It is also the source of the fact that $\partial\mathcal{E}_X$ is locally polyhedral away from the modified nef cone, with extremal rays generated by exceptional prime divisors.

2.3.3 Pull-back and push-forward of pseudoeffective classes

Proposition 2.35. *Let $\pi\colon \hat{X} \to X$ be a proper bimeromorphic morphism of compact Kähler varieties. Then $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is pseudoeffective if and only if $\pi^*\alpha$ is, and π_* carries pseudoeffective classes to pseudoeffective classes.*

Proof. If $T \in \alpha$ is closed positive with local potentials, then π^*T is defined by pulling back the potentials, is again closed positive with local potentials, and lies in $\pi^*\alpha$. Conversely, if $\hat{T} \in \pi^*\alpha$ is closed positive, then $\pi_*\hat{T}$ is closed and positive — push-forward of currents preserves both — and its class is $\pi_*\pi^*\alpha = \alpha$, since π is bimeromorphic. The same computation gives the last statement. ∎

Remark 2.36. The one point requiring care is that on a singular space the push-forward of a current with local potentials need not have local potentials; the statement above is about classes in $H^{1,1}_{\mathrm{BC}}$, where Proposition 1.4 supplies a representative with local potentials on either side.

An alternative and often more convenient proof of both directions: by Proposition 2.25, α is pseudoeffective iff $\alpha + \epsilon\{\omega\}$ is big for every $\epsilon > 0$, and bigness is a two-sided bimeromorphic invariant by Proposition 2.26.

Fact 2.37. Pseudoeffectivity is preserved under small deformation — in contrast to Kählerness of a singular space.

Remark 2.38. Pseudoeffectivity is the hypothesis in almost every theorem of these notes — Boucksom's nefness criterion (Proposition 2.46), the divisorial Zariski decomposition, the BDPP circle (Section 3.2). The reason is structural: it is exactly the condition under which the class contains a current with minimal singularities, and each of those theorems is read off from that current.

2.4 Nefness

Nefness is the terminal condition of the programme: a minimal model is by definition one on which $K_X + B$ is nef, so every step of the MMP is an attempt to manufacture nefness, and every criterion below is a way of certifying that it has been achieved. It is also the tier where the transcendental theory diverges most sharply from the projective one.

2.4.1 The definition

Definition 2.39 ([HP16, Definition 3.10]; [Pǒ98])**.** Let X be an irreducible reduced compact complex space and $u \in H^{1,1}_{\mathrm{BC}}(X)$ a class represented by a form α with local potentials. Then u is **nef** if for some positive $(1,1)$-form ω on X and every $\epsilon > 0$ there is $f_\epsilon \in \mathcal{A}^0(X)$ with

$$\alpha + i\partial\bar\partial f_\epsilon \;\geq\; -\epsilon\,\omega.$$

In line bundle terms: L is nef if for every $\epsilon > 0$ there is a smooth Hermitian metric h_ϵ with $\Theta_{h_\epsilon} \geq -\epsilon\omega$.

Warning. The definition demands a smooth f_ϵ. This is precisely why Demailly regularization is needed at almost every step where nefness is verified — see the proof of Proposition 2.46.

Relative version ([DHY23]): β is relatively nef if $\beta + \omega$ is relatively Kähler for every relatively Kähler ω.

2.4.2 Basic properties

Proposition 2.40. *Let X be compact Kähler with a fixed Kähler form ω.*

1. *A class α is nef if and only if $\alpha + \epsilon\{\omega\}$ is Kähler for every $\epsilon > 0$; equivalently, iff α is a limit of Kähler classes.*
2. *$\mathrm{Nef}(X)$ is a closed convex cone containing no line, and $\mathrm{Nef}(X) + \mathcal{K}_X \subseteq \mathcal{K}_X$.*
3. *Nef $\Rightarrow$ pseudoeffective.*

Proof. (1) If α is nef then $\alpha + \epsilon\{\omega\}$ is Kähler by Lemma 2.5. Conversely, if $\alpha + \epsilon\{\omega\}$ is Kähler for every $\epsilon > 0$, pick a Kähler form $\alpha + \epsilon\omega + i\partial\bar\partial f_\epsilon > 0$ in that class; then $\alpha + i\partial\bar\partial f_\epsilon \geq -\epsilon\omega$, which is Definition 2.39. The reformulation as a limit is immediate.
(2) Convexity and the cone property follow from (1), since the Kähler cone is convex; closedness follows from (1) as well, a limit of limits of Kähler classes being one. Salience follows from that of $\mathcal{K}_X$ (Proposition 2.3) by passing to closures, using $\int_X \alpha^n \geq 0$ for nef α. The last inclusion is Lemma 2.5.

(3) By (1), $\alpha + \epsilon\{\omega\}$ is Kähler, hence pseudoeffective, for all $\epsilon > 0$. Letting $\epsilon \to 0$ and using that $\mathcal{E}_X$ is closed (Proposition 2.32) gives $\alpha \in \mathcal{E}_X$. ∎

Together with Proposition 2.25 this yields the skeleton of these notes:

$$\mathcal{K}_X \subset \mathrm{Nef}(X) \subset \mathcal{E}_X, \qquad \mathcal{K}_X \subset \mathcal{B}_X \subset \mathcal{E}_X, \qquad \mathrm{Nef}(X) = \overline{\mathcal{K}_X}, \quad \mathcal{E}_X = \overline{\mathcal{B}_X}.$$

Remark 2.41. Nef is a *closed* condition, Kähler an *open* one, and this is why the two are used for different jobs. Kählerness survives perturbation, so it is what one carries along during a construction; nefness survives passage to the limit, so it is what one concludes at the end of one. The base point free theorem (Theorem 2.52) is the machine that converts the closed condition back into geometry.

2.4.3 Metric versus numerical nefness

This is the single most important place where projective intuition breaks.

Definition 2.42. α is **metrically nef** if Definition 2.39 holds, and **numerically nef** if $\alpha \cdot C \geq 0$ for every irreducible curve $C \subset X$.

Fact 2.43. On a compact Kähler variety, metrically nef $\Rightarrow$ numerically nef. The converse is false.

Example 2.44 ([DH25, p. 8])**.** Let X be a compact Kähler surface of algebraic dimension one, with elliptic fibration $\varphi\colon X \to B$ onto a projective curve B. Take H ample on B and set $u := -\varphi^* H$, so u is *anti*-nef. Every curve $C \subset X$ is contracted by φ, hence $u \cdot C = 0$ for all C. So u is numerically nef and metrically anti-nef simultaneously.

Remark 2.45. The moral is that on a non-algebraic Kähler space *curves see too little*. This is exactly why $\overline{\mathrm{NE}}(X)$ must be enlarged to $\overline{\mathrm{NA}}(X)$ in Section 3. The two notions do agree on algebraic and Moishezon varieties, and — crucially for the MMP — for $K_X + B$ on a Kähler 3-fold under a pseudoeffectivity hypothesis. That last statement is Step 1 of the Das–Hacon proof of the cone theorem for Kähler 3-fold dlt pairs: the argument needs metric and numerical nefness to coincide *for the class* $K_X + B$, via Proposition 2.46.

2.4.4 Boucksom's nefness criterion

Write $\nu(\alpha, x)$ for the minimal multiplicity (generic Lelong number) of α at x.

Proposition 2.46 ([Bou04, Proposition 3.2])**.** *Let $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ be pseudoeffective. Then*

1. *α is nef $\iff$ $\nu(\alpha, x) = 0$ for every $x \in X$;*
2. *α is modified nef $\iff$ $\nu(\alpha, D) = 0$ for every prime divisor D.*

Proof. ($\Rightarrow$) If α is nef, then for each $\epsilon > 0$ the smooth form $\theta_\epsilon := \alpha + i\partial\bar\partial f_\epsilon$ is an almost positive current in the class α with $\theta_\epsilon \geq -\epsilon\omega$. A smooth form has vanishing Lelong number at every point, so the infimum over ϵ defining $\nu(\alpha, x)$ is 0.
($\Leftarrow$) Suppose $\nu(\alpha, x) = 0$ for all x. For each $\epsilon > 0$ let $T_{\min,\epsilon}$ be a current with minimal singularities in the class $\alpha[-\epsilon\omega]$, i.e. among closed currents $T \in \alpha$ with $T \geq -\epsilon\omega$; such a current exists because α is pseudoeffective. Minimal multiplicity is computed by these currents, so $\nu(T_{\min,\epsilon}, x) = 0$ for every x. Applying Demailly regularization [Dem92] to $T_{\min,\epsilon}$ replaces it by a *smooth* form in α with lower bound $-\epsilon'\omega$ for any $\epsilon' > \epsilon$, the loss of positivity being controlled by the Lelong numbers, which vanish. Since $\epsilon' > 0$ was arbitrary this is exactly Definition 2.39. ∎

Remark 2.47. The criterion converts nefness — a statement about the existence of infinitely many smooth approximating potentials — into a statement about *one number at each point*, computed from the class alone. Every verification of nefness that does not go through the cone (Theorem 4.4) goes through this. The second clause is what defines the "modified" tier, and its projective shadow is the statement that a pseudoeffective divisor is nef iff its asymptotic vanishing order is 0 along every divisorial valuation.

2.4.5 Pull-back and push-forward of nef classes

Proposition 2.48. *Let $f : X \to Y$ be a surjective morphism of compact Kähler varieties and $\alpha \in H^{1,1}_{\mathrm{BC}}(Y)$.*

1. *If α is nef then $f^*\alpha$ is nef.*
2. *Conversely, if $f^*\alpha$ is nef then α is nef [Pǎ98].*
3. *Restriction to a subvariety preserves nefness (Proposition 2.28).*

The push-forward of a nef class need not *be nef.*

Proof of (1). Fix positive forms ω_Y on Y and ω_X on X. Given $\epsilon > 0$, Definition 2.39 supplies a smooth f_ϵ on Y with $\alpha + i\partial\bar\partial f_\epsilon \geq -\epsilon\,\omega_Y$. Pulling back, and using that pull-back preserves the order relation between real $(1,1)$-forms,

$$f^*\alpha + i\partial\bar\partial(f_\epsilon \circ f) \;\geq\; -\epsilon\, f^*\omega_Y.$$

Since X is compact and $f^*\omega_Y$ is a semi-positive form, there is a constant $C > 0$, independent of ϵ, with $f^*\omega_Y \leq C\,\omega_X$. Hence $f^*\alpha + i\partial\bar\partial(f_\epsilon \circ f) \geq -\epsilon C\,\omega_X$. As $\epsilon > 0$ was arbitrary and $f_\epsilon \circ f$ is smooth, this is Definition 2.39 for $f^*\alpha$. ∎

Remark 2.49. Direction (2) is genuinely harder and is Păun's theorem [Pă98]: it is the statement that numerical effectiveness descends along surjections, and it has no one-line proof of the above kind.

This is the mirror image of Proposition 2.26: nefness pulls back well and pushes forward badly, bigness the reverse. Note also the contrast with bigness in Section 2.2.4: nefness restricts to *every* subvariety, bigness only to those not contained in the degeneracy locus. The one positive statement in the push-forward direction is that the negative part of the push-forward of a nef class under a bimeromorphic morphism vanishes — so nefness is not destroyed *divisorially* by an MMP step, even though it is not preserved outright.

Warning. In the MMP for Kähler generalized pairs one frequently needs a class to be *NQC* (a non-negative combination of nef classes of the appropriate type) rather than merely nef, because the transcendental cone can fail to be spanned by the classes the induction runs on [DHY23].

2.5 Semi-positivity and semi-ampleness

2.5.1 The implication chain

Proposition 2.50. *For a class on a compact Kähler variety,*

$$\textit{Kähler} \Longrightarrow \textit{semi-ample} \Longrightarrow \textit{semi-positive} \Longrightarrow \textit{nef} \Longrightarrow \textit{pseudoeffective}.$$

Proof. Semi-ample ⇒ semi-positive: a semi-ample class is $f^*\omega_Y$ for a Kähler ω_Y downstairs, and the pull-back of a positive form is semi-positive (Fact 2.12). Semi-positive ⇒ nef: a semi-positive representative satisfies Definition 2.39 with $f_\epsilon = 0$. Nef ⇒ pseudoeffective is Proposition 2.40(3). Kähler ⇒ semi-ample is the case $Y = X$, $f = \mathrm{id}$. ∎

Warning. Nef ⇏ semi-positive. This failure has no projective counterpart worth the name, and it is why nefness is a genuinely weaker and harder condition here: a nef class need not contain *any* smooth semi-positive representative, the best available being the family $\alpha + \epsilon\omega$ of Proposition 2.40(1). Nef ⇏ semi-ample is already false projectively — that is the abundance problem — and transcendentally there is not even a linear system in which to pose the question.

Two supplementary facts of the same kind, useful but not load-bearing here: a semi-ample and strictly nef class is Kähler; and a log canonical class numerically equal to the pull-back of a Kähler class is semi-ample.

Remark 2.51. Semi-ampleness is what turns a *class* into a *morphism*. Nefness certifies that the MMP has stopped; semi-ampleness is what lets one contract, take the Iitaka fibration, or descend the class to a base. This is why the transcendental substitute below is stated as the existence of a fibration f with $\alpha = f^*\omega_Y$, and not as any statement about sections.

2.5.2 The base point free theorem

In the transcendental setting there is no linear system, so "semi-ample" cannot mean "$|mL|$ is base point free". The correct replacement is the existence of a semi-ample fibration.

Theorem 2.52 ([DHP24, Theorem 5.5]). *Let (X, B) be a compact Kähler 3-fold klt pair and $\alpha \in N^1(X)$ a nef class such that $\alpha - (K_X + B)$ is nef and big. Then there is a proper morphism $f : X \to Y$ with connected fibres onto a normal compact Kähler variety Y with rational singularities, and a Kähler class $\omega_Y \in N^1(Y)$, such that $\alpha = f^*\omega_Y$. If $\alpha - (K_X + B)$ is Kähler, then f is projective.*

The dlt version is [DHP24, Corollary 5.6]; the $\mathbb{Q}$-factorial version is in [DH25]; the generalized pair version is [DHY23, Theorem 3.14]. All of these are stated for 3-folds.

The dimension restriction has since been removed. The base point free conjecture in the Kähler setting — equivalently Tosatti's conjecture — is now a theorem of Hacon–Xie in arbitrary dimension.

Theorem 2.53 (Transcendental base point free theorem, [HX26, Theorem 1.4]). *Let $(X, B + \beta)$ be a compact Kähler gklt pair such that $K_X + B + \beta_X$ is nef and $B + \beta_X$ is modified big. Then $K_X + B + \beta_X$ is semiample: there is a Moishezon contraction $f : X \to Y$ onto a compact Kähler space and a Kähler class ω_Y on Y with*

$$K_X + B + \beta_X \equiv f^*\omega_Y.$$

If X is strongly $\mathbb{Q}$-factorial, then f is projective.

Idea of proof. The argument splits on whether $K_X + B + \beta_X$ is big. In the non-big case one uses Ou's theorem (Section 3.2) to produce an MRC fibration, runs a relative MMP to a Mori fibre space, and descends by the canonical bundle formula, concluding by induction on the dimension of the base. In the big case one uses that the non-nef locus coincides with the null locus, hence is analytic, places a Kähler current with weak analytic singularities along it, and runs an induction over the nested multiplier-ideal subspaces; the resulting morphisms are glued on a formal neighbourhood of $E_+(\alpha)$ and then algebraized to a global morphism. The proof is the subject of a separate note

in this series and is not reproduced here.

Remark 2.54. The statement is about a *class*, not a line bundle, and there is no linear system anywhere in it: "semiample" means $\equiv f^*\omega_Y$ for a morphism f and a Kähler class ω_Y. That reformulation, forced by the absence of divisors, is the whole reason the transcendental statement is harder than its projective counterpart.

3 Mori cone, generalized Mori cone, and duality of cones

Everything so far has been about cones in $N^1(X)$. This section introduces the cones in $N_1(X)$ that they are tested against, and the duality between the two sides — which turns out to be the same statement as the Kählerness criterion of the appendix.

3.1 The two Mori cones

Definition 3.1 ([HP16, p. 226]; [DH25, Definition 2.2])**.**
1. $\overline{\mathrm{NA}}(X) \subset N_1(X)$ is the closed cone generated by the classes of *positive closed currents*.
2. $\overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NA}}(X)$ is the closure of the cone generated by the currents of integration T_C over irreducible curves $C \subset X$.

Proposition 3.2. *The inclusion $\overline{\mathrm{NE}}(X) \subseteq \overline{\mathrm{NA}}(X)$ is in general strict, already for projective X.*

Proof. $\overline{\mathrm{NE}}(X)$ spans a space of dimension $\rho(X)$, while $\overline{\mathrm{NA}}(X)$ spans $H^{n-1,n-1}(X)$. For a projective K3 surface the Torelli theorem gives $\rho(X) \leq 20 = h^{1,1}(X)$, so $\overline{\mathrm{NE}}(X)$ lies in a proper subspace $N_1(X) \subsetneq H_{1,1}(X,\mathbb{R})$ while $\overline{\mathrm{NA}}(X)$ contains classes outside it. More dramatically, there are K3 surfaces of Picard number 0: no curves at all, yet plenty of positive closed currents. ∎

Remark 3.3. This is the cone-side shadow of Example 2.44: curves are not enough to detect positivity, and $\overline{\mathrm{NA}}(X)$ is the smallest enlargement that fixes it. Extremality is not inherited between the two cones, so one must be careful which cone an extremal ray argument is run in.

Fact 3.4 (Generators)**.** $\overline{\mathrm{NA}}(X)$ is generated by the functionals $\int_V (-) \wedge \omega^{d-1}$ over subvarieties V. This is what makes Theorem 4.1 applicable to it, and is the reason cone theorem proofs need only check generators of the shape $[\hat{\omega}]^2$, $[\hat{\omega}] \cdot [\hat{S}]$, $[\hat{C}]$.

$\overline{\mathrm{NA}}(X)$ contains no line, which is what makes extremal-ray arguments possible; the push-forward of $\overline{\mathrm{NA}}$ under a bimeromorphic morphism is again $\overline{\mathrm{NA}}$; and contracting a negative extremal ray of $\overline{\mathrm{NA}}$ preserves the Kähler condition. The K_X-negative structure of $\overline{\mathrm{NA}}(X)$ — the cone theorem itself — is the subject of a separate note in this series and is only invoked here.

3.2 Duality of cones

3.2.1 $\mathrm{Nef}(X)$ and $\overline{\mathrm{NA}}(X)$

Theorem 3.5 ([HP16, Proposition 3.15]; [DHP24, Proposition 2.4]; [DP04, Corollary 0.4])**.** *Let X be a normal compact Kähler variety with rational singularities. Then $\mathrm{Nef}(X)$ and $\overline{\mathrm{NA}}(X)$ are dual to each other under the isomorphism $\Phi\colon N^1(X) \to N_1(X)^*$ of Proposition 1.12. In particular $\omega \in H^{1,1}_{\mathrm{BC}}(X)$ is Kähler if and only if $\omega \cdot \gamma > 0$ for every $\gamma \in \overline{\mathrm{NA}}(X) \setminus \{0\}$.*

The second sentence is exactly the Kleiman-type criterion Theorem 4.4: duality and the Kählerness criterion are two faces of one statement. The easy half — that a Kähler class is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$ — is Corollary 2.9. The projective baseline for comparison is $\mathrm{Nef}(X)^* = \overline{\mathrm{NE}}(X)$; the manifold version, $\mathcal{N}^* = \overline{\mathcal{K}}$ together with $\mathrm{Eff}^*(X) = \mathrm{Nef}(X)$, is [Dem12, 19.6].

3.2.2 Pseudoeffective versus movable: BDPP

Theorem 3.6 ([BDPP13], projective case)**.** *On a projective manifold the pseudoeffective cone $\mathcal{E}$ is dual to the movable cone $\mathcal{M}$: a class is pseudoeffective iff it has non-negative intersection with every curve moving in a family covering X.*

Warning. Two statements are both called "BDPP" and must be kept apart.

(a) *Cone duality.* [BDPP13, Conjecture 2.3]: for an arbitrary compact Kähler manifold, $\mathcal{E}$ and $\mathcal{M}$ are dual. This is still open in the Kähler case.

(b) *Uniruledness.* K_X is pseudoeffective iff X is not uniruled. For compact Kähler manifolds this is now a theorem of Ou [Ou25].

It is (b) that Hacon–Păun assume in order to deduce the cone theorem in arbitrary dimension. Conflating (a) with (b) misstates what is known.

Statement (b) is proved in full in a separate note of this series, following Ou [Ou25]; the projective case of the duality statement, following [BDPP13], is treated there as well. We use both freely below.

Other dualities on the same circle: the cone of nef curves is dual to the pseudoeffective cone; Serre duality induces a pairing between the transcendental pseudoeffective and mobile cones, and between the nef cone and the cone of effective $(n-1,n-1)$-classes.

4 Appendix: Kählerness criterion

Three criteria of increasing usability inside the MMP: an intersection-theoretic one, a topological refinement, and a duality one. All are stated without proof; the proof ideas are recorded because they are what gets reused.

4.1 The Demailly–Păun Nakai–Moishezon criterion

Theorem 4.1 ([DP04, Corollary 2.39]; singular version in [DHP24]). *Let X be a compact normal Kähler variety, ω a Kähler form and $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$. Then*

1. *α is nef $\iff$ $\int_V \alpha^k \wedge \omega^{p-k} \geq 0$ for every analytic p-dimensional $V \subset X$ and all $0 < k \leq p$;*
2. *α is Kähler $\iff$ the same integrals are all > 0.*

There is a variant tailored to MMP applications: starting from a smooth ∂- and $\bar{\partial}$-closed $(1,1)$-form α with $\{\alpha\}$ nef and big and $\int_{V_{\mathrm{reg}}} \alpha^{\dim V} > 0$ for every positive-dimensional reduced analytic V, one concludes that α is Kähler [DHP24, Theorem 2.29].

> *big + nef ≃ singular Kähler metric (Kähler current).*

By Demailly regularization the Kähler current can be taken with analytic singularities, i.e. it is an honest smooth Kähler metric on a Zariski open subset.

4.2 The Kähler cone is a connected component of the positive cone

Theorem 4.2 ([DP04, Main Theorem 0.1]; [Dem12, 18.5]). *Let X be a compact Kähler manifold and*

$$\mathcal{P} := \left\{ [\alpha] \in H^{1,1}(X,\mathbb{R}) \;\middle|\; \int_Y \alpha^{\dim Y} > 0 \text{ for all irreducible analytic } Y \subset X,\ \dim Y > 0 \right\}.$$

Then $\mathcal{K}$ is one of the connected components *of $\mathcal{P}$. If X is projective, $\mathcal{K} = \mathcal{P}$.*

Example 4.3 (Generic torus: why "one of the components"). Let $X = \mathbb{C}^n/\Lambda$ be a generic complex torus, which has no analytic subsets other than points and X. Numerical

positivity then reduces to the single condition $\int_X \alpha^n > 0$, and $(1,1)$-classes correspond to constant Hermitian forms on $\mathbb{C}^n$. So $\mathcal{P} = \{\det(\alpha) > 0\}$, and the theorem becomes the linear algebra statement that the positive definite forms are *one* connected component — the others being forms of signature (p,q) with $p+q=n$ and q even. These are not even convex when $p, q > 0$.

4.3 The Kleiman-type criterion

Theorem 4.4 ([HP16, Theorems 3.17–3.18])**.** *Let X be a normal compact Kähler 3-fold with rational singularities and $[\eta] \in N^1(X)$. Then $[\eta]$ is Kähler if and only if $[\eta]$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$.*

Idea of proof. Take a desingularization $\pi\colon \hat{X} \to X$ which is projective with SNC exceptional locus, and an effective π-exceptional D with $-D$ π-ample. For $N \gg 0$ the class $[N\pi^*\eta - D]$ is strictly positive on $\overline{\mathrm{NA}}(\hat{X})$, so by Theorem 4.1 it is a Kähler class $[\hat{\omega}]$. Pushing forward, $\pi_*(\hat{\omega}) = N\eta + \partial\bar{\partial}R$ is a Kähler current on X, smooth outside the image E of the exceptional locus. A Kähler current contains a Kähler form iff its restriction to each component Z of the Lelong level sets is Kähler; here dimension three enters: $Z \subseteq E$ forces Z to be a point or a curve, and on a curve the positivity hypothesis $T_Z(\eta) > 0$ says exactly that $i_Z^*\eta$ has positive degree.

Warning. Two hypotheses cannot be dropped. Dimension 3 is used to force Lelong level sets to be points or curves. Rational singularities is needed for the duality of Theorem 3.5 to hold at all. Whether a Kleiman criterion holds in higher dimension is open.

4.4 Fujiki, Moishezon, projective

Theorem 4.5 ([Dem12, 14.17]; [DP04, Theorem 3.4]; [Bou04, Proposition 2.3])**.** *A compact complex* manifold *X is bimeromorphic to a Kähler manifold, i.e. $X \in \mathcal{C}$, if and only if it carries a Kähler current — equivalently, iff the big cone is non-empty.*

Idea of proof. If X is bimeromorphic to a Kähler Y, resolve to a modification $\mu\colon \tilde{Y} \to X$ and push forward a Kähler form: compactness gives $\mu^*\omega \le C\tilde{\omega}$, so $T = \mu_*\tilde{\omega} \ge C^{-1}\omega$ is a Kähler current. Conversely, given a Kähler current T, Demailly regularization replaces it by T' with analytic singularities; on a resolution $\mu^*T' = \lambda[\tilde{D}] + \tilde{\alpha}$ with $\tilde{\alpha}$ smooth and $\tilde{\alpha} \ge \epsilon\mu^*\omega$, and subtracting small multiples of exceptional classes blow-up

by blow-up produces an honest Kähler form.

Warning. Per [DH24, Remark 2.3] it is *not known* whether a singular Fujiki class $\mathscr{C}$ variety carries a big $(1,1)$-class. The obstruction is that the Poincaré lemma may fail on a singular space, so the push-forward of a Kähler current need not have local potentials.

Key takeaways

- Positivity lives on $H^{1,1}_{\mathrm{BC}}(X)$, not on divisors; rational singularities is what makes $H^{1,1}_{\mathrm{BC}}(X) \hookrightarrow H^2(X,\mathbb{R})$ and hence intersection theory available.
- Kähler and big are *open* conditions; nef and pseudoeffective are *closed* ones. $\mathrm{Nef} = \overline{\mathcal{K}}$, $\mathcal{K} = \operatorname{int}\mathrm{Nef}$, $\mathcal{E} = \overline{\mathcal{B}}$, $\mathcal{B} = \operatorname{int}\mathcal{E}$.
- Each tier absorbs the weaker one on its own side: Kähler + nef = Kähler, big + pseudoeffective = big.
- Bigness is a two-sided bimeromorphic invariant; nefness pulls back well but does not push forward.
- Nefness restricts to *every* subvariety; bigness restricts only to those avoiding the locus where the Kähler current degenerates, which is why one restricts to *general* fibres.
- Metric nefness is strictly stronger than numerical nefness on a non-algebraic space — curves see too little — which is why $\overline{\mathrm{NE}}(X)$ must be enlarged to $\overline{\mathrm{NA}}(X)$.
- $\mathrm{Nef}(X)$ and $\overline{\mathrm{NA}}(X)$ are dual, and that duality *is* the Kählerness criterion.

References

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[Bou02] S. Boucksom, *On the volume of a line bundle*, International Journal of Mathematics **13** (2002), no. 10, 1043–1063.

[Bou04] Sébastien Boucksom, *Divisorial zariski decompositions on compact complex manifolds*, Annales scientifiques de l'Ecole normale supérieure, vol. 37, 2004, pp. 45–76.

[Dem92] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), no. 3, 361–409.

[Dem12] _____, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, Higher Education Press, Beijing, 2012.

[DH24] O. Das and C.D. Hacon, *On the minimal model program for Kähler 3-folds*, 2024.

[DH25] Omprokash Das and Christopher Hacon, *The log minimal model program for Kähler 3-folds*, J. Differential Geom. **130** (2025), no. 1, 151–207.

[DHP24] O. Das, C.D. Hacon, and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68.

[DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.

[DP04] J.-P. Demailly and M. Păun, *Numerical characterization of the kähler cone of a compact kähler manifold*, Annals of mathematics (2004), 1247–1274.

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.

[HX26] C.D. Hacon and L. Xie, *On the Kähler MMP and the transcendental base-point-free theorem*, 2026.

[Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

[Pŏ98] Mihai Păun, *Sur l'effectivité numérique des images inverses de fibrés en droites*, Math. Ann. **310** (1998), no. 3, 411–421.

The Transcendental Volume Function

Kähler MMP Yi Li

The aim of this note is to introduce the transcendental volume.

Contents

1 Definition of volumes 1

2 Fujita's approximation 3

3 Basic properties of volume functions 4

3.1 For (ℝ-)divisors 4

3.2 For cohomology classes 5

4 Upper semi-continuity of volumes 5

5 Deformation invariance of adjoint volumes 6

6 Volume in divisorial Zariski decomposition 6

7 Volume in the minimal model program 7

8 Jiao's proof of upper semi-continuity of volume function 7

1 Definition of volumes

Definition 1.1 (Volume of a line bundle, [Laz04, Definition 2.2.31]). Let X be an irreducible projective variety of dimension n, and let L be a line bundle on X. The volume of L is defined to be the non-negative real number

$$\mathrm{vol}(L) = \mathrm{vol}_X(L) = \limsup_{m\to\infty} \frac{h^0\left(X, L^{\otimes m}\right)}{m^n/n!}.$$

The volume $\mathrm{vol}(D) = \mathrm{vol}_X(D)$ of a Cartier divisor D is defined similarly, or by passing to $\mathcal{O}_X(D)$.

Definition 1.2 (Volume of a $\mathbb{R}$-divisor, [FKL16, p. 8])**.** If X is a normal projective variety, D is an $\mathbb{R}$-divisor and $n = \dim X$, then we define the volume of D by

$$\mathrm{vol}(X, D) = \limsup_{m\to\infty} \frac{n!\, h^0(X, mD)}{m^n}$$

where $h^0(X, mD) := h^0(X, \lfloor mD \rfloor)$.

Definition 1.3 (Volume of cohomology classes, [Bou02])**.** Let X be a compact Kähler n-fold. We define the volume of a cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ by

$$\mathrm{vol}(\alpha) := \sup_{T\in\alpha} \int_X T_{ac}^n = \sup_{T\in\alpha} \int_{X\setminus \mathrm{Sing}(T)} T^n$$

for T ranging over closed positive $(1,1)$-currents in α, in case α is pseudo-effective. If it is not, we set $\mathrm{vol}(\alpha) = 0$.

Remark 1.4 (Lebesgue's decomposition and absolutely continuous part)**.** A current T on an n-fold X can be seen as a form with distribution coefficients. When T is positive, the distributions in question are positive measures which admit a Lebesgue decomposition into an absolutely continuous part (with respect to the Lebesgue measure on X) and a singular part.
We therefore get a decomposition of T itself into an absolutely continuous part T_{ac} and a singular part T_{sing}. The decomposition

$$T = T_{ac} + T_{sing}.$$

The reason to take the absolutely continuous part is that the n-fold product of an arbitrary closed positive current T is not well defined (it requires a locally bounded potential). Once we assume T has locally bounded potentials, the product T_{ac}^n is well defined via Bedford–Taylor theory.
However, the decomposition does not behave well at the cohomological level. As a closed positive current T, the absolutely continuous part T_{ac} need not be closed.

Remark 1.5 (Volume on normal varieties)**.** On a compact normal Kähler variety, we can first take a resolution $\pi : X' \to X$, and define the volume of a class $\alpha \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ to be

$$\mathrm{vol}(\alpha) = \sup_{[T]=\pi^*\alpha} \int_{X'} T_{ac}^n.$$

It is well defined, since the volume function for a smooth variety is a birational invariant (see Theorem 3.4).

One can show that the analytically defined volume coincides with the algebraically defined volume.

Theorem 1.6 ([Bou02, Theorem 1.2]). *Let L be a pseudoeffective line bundle on a compact Kähler manifold X. Then the volume of L satisfies*

$$\mathrm{vol}(L) = \max_{T} \int_X T_{ac}^n$$

for T ranging among the closed positive $(1,1)$-currents in the cohomology class $c_1(L)$.

2 Fujita's approximation

One of the central properties that the volume function satisfies is the Fujita approximation. Both big line bundles and big cohomology classes satisfy this property.

Theorem 2.1 (Fujita approximation (for big line bundle on projective variety)). *Let L be a big line bundle on a projective manifold X. Then, for every $\varepsilon > 0$, there exist a modification $\mu : \widetilde{X} \to X$, an ample $\mathbb{Q}$-line bundle A, and an effective $\mathbb{Q}$-divisor E on $\widetilde{X}$ (all depending on ε) such that:*

(i) $\mu^* L \sim_{\mathbb{Q}} A + E$ *on* $\widetilde{X}$,

(ii) $|\mathrm{vol}(A) - \mathrm{vol}(L)| < \varepsilon$, *in particular* $A^n > \mathrm{vol}(L) - \varepsilon$.

Theorem 2.2 (Fujita approximation (for big cohomology classes on compact Kähler manifold), [Bou02, Theorem 1.4]). *Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class on X. Then, for every $\varepsilon > 0$, there exist a modification $\mu : \widetilde{X} \to X$, a Kähler class ω, and an effective real divisor D on $\widetilde{X}$ such that:*

(i) $\mu^\star \alpha = \omega + \{D\}$ *as cohomology classes,*

(ii) $|\mathrm{vol}(\alpha) - \mathrm{vol}(\omega)| < \varepsilon$.

Idea of proof. Regularize positive currents in a big class by currents with analytic singularities, then resolve their ideals. Upstairs the divisorial singularities become an effective divisor, leaving a smooth positive contribution. A small exceptional correction makes the moving class Kähler. Controlling the lost mass gives the approximation of volume; that mass estimate is the substantive content of Fujita approximation.

3 Basic properties of volume functions

3.1 For ($\mathbb{R}$-)divisors

Volume depends only on the numerical class of an $\mathbb{R}$-divisor.

Proposition 3.1 (Volume is numerical invariant, [FKL16, Theorem 3.5]). *Let D be an $\mathbb{R}$-divisor on a proper normal variety X of dimension n. If D' is an $\mathbb{R}$-divisor on X such that $D' - D$ is a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then* $\mathrm{vol}(D) = \mathrm{vol}(D')$.

The volume function is continuous on $N^1(X)_{\mathbb{R}}$.

Proposition 3.2 (Continuity of volume function, [Laz04, Corollary 2.2.45]). *The function $\xi \mapsto \mathrm{vol}(\xi)$ on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a homogeneous continuous function*

$$vol : N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

Volume increases in effective directions.

Proposition 3.3 (Volume increases in effective directions, [Laz04, Example 2.2.48]). *If $\xi \in N^1(X)_{\mathbb{R}}$ is big and $e \in N^1(X)_{\mathbb{R}}$ is effective, then*

$$\mathrm{vol}(\xi) \leq \mathrm{vol}(\xi + e).$$

The volume satisfies the birational invariance property.

Theorem 3.4. *Let*

$$\nu : X' \longrightarrow X$$

be a birational projective morphism of irreducible normal projective varieties. Then

$$\mathrm{vol}_X(\xi) = \mathrm{vol}_{X'}(\nu^*\xi)$$

for any class $\xi \in N^1(X)_{\mathbb{R}}$.

For a generically finite morphism we have a similar pullback formula.

Theorem 3.5. *Let $f : Y \to X$ be a proper, dominant, generically finite morphism of normal projective varieties over k. For any Cartier divisor D on X, we have*

$$\mathrm{vol}_Y\left(f^*D\right) = \deg(f)\,\mathrm{vol}_X(D).$$

Analogously to the case of global sections, the volume function satisfies the following property.

Theorem 3.6. *Let $g : X \to Y$ be a birational morphism of normal projective varieties, let D be an $\mathbb{R}$-Cartier divisor on Y, and let G be an $\mathbb{R}$-divisor on X. If $G - g^*D \geq 0$ is effective and g-exceptional, then*

$$\mathrm{vol}(X, G) = \mathrm{vol}(Y, D).$$

3.2 For cohomology classes

The volume function is continuous on the Bott–Chern cohomology space.

Proposition 3.7 ([Bou02, Corollary 4.11])**.** *The volume* $\mathrm{vol} : H^{1,1}(X, \mathbb{R}) \to \mathbb{R}$ *is a continuous function.*

The volume function is log-concave.

Proposition 3.8 (Log concavity of volume function)**.** *Let X be a compact Kähler manifold. If $\alpha, \alpha' \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ are big classes, then the volume satisfies the log concavity property*

$$\mathrm{vol}(\alpha + \alpha')^{1/n} \geq \mathrm{vol}(\alpha)^{1/n} + \mathrm{vol}(\alpha')^{1/n}.$$

As for $\mathbb{R}$-divisors, volume is birationally invariant for cohomology classes.

Proposition 3.9. *Let*

$$\nu : X' \longrightarrow X$$

be a birational morphism of irreducible compact Kähler manifolds. Then

$$\mathrm{vol}_X(\xi) = \mathrm{vol}_{X'}(\nu^*\xi)$$

for any class $\xi \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$.

4 Upper semi-continuity of volumes

Jiao proved the following upper semi-continuity of volume for a projective family with reduced irreducible fibers.

Theorem 4.1 ([Jia25, Theorem 1.2])**.** *Let $f : X \to T$ be a projective flat morphism of varieties over $\mathbb{C}$ with reduced and irreducible fibers and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then for any $a \in \mathbb{R}$, the set*

$$\{t \in T \mid \mathrm{vol}\,(D_t) < a\}$$

is open in T.

Boucksom proved the following upper semi-continuity of volume for smooth Kähler families.

Theorem 4.2 ([Bou02, Proposition 4.13]). *The volume is upper-semicontinuous in Kähler deformations in the following sense: if $\mathcal{X} \to S$ is a deformation of Kähler n-folds and $\alpha \in H^{1,1}(X, \mathbb{R})$ is a pseudoeffective class on the central fibre $X_0 =: X$, then one has*

$$v(\alpha) \geq \limsup_{k \to +\infty} v(\alpha_k)$$

for every sequence $\alpha_k \in H^{1,1}\left(X_{t_k}, \mathbb{R}\right)$ converging to α (with $t_k \to 0$).

5 Deformation invariance of adjoint volumes

The volume of log canonical divisors or adjoint classes satisfies deformation invariance properties (the proof will be given in a separate note).

Theorem 5.1 ([HMX18, Corollary 4.3]). *Let $\pi : X \longrightarrow T$ be a projective morphism of smooth varieties. Suppose that (X, Δ) is log canonical and has simple normal crossings over T. Then the volume function*

$$t \longmapsto \mathrm{vol}(X_t, K_{X_t} + \Delta_t)$$

is independent of t.

This is also true for generalized Kähler pairs, when the central fiber is projective with a big adjoint class.

Theorem 5.2 (HLR25'). *Let $f : X \to S$ be a smooth family from a Kähler manifold X onto a smooth, connected, relatively compact curve S. Fix a point $0 \in S$ and assume that $(X, B + \beta)$ is a generalized klt pair such that (X, B) is log smooth over S and $\beta = \overline{\beta}$ where β is nef over S and X_0 is projective. If $K_{X_0} + B_0 + \beta_0$ is big, then the volume function*

$$t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \beta_t)$$

is constant on some neighborhood $U \subset S$ of 0.

6 Volume in divisorial Zariski decomposition

We have already seen that the volume may increase in the effective direction; in this section we show that subtracting the negative part of the Zariski decomposition does not change the volume.

7 Volume in the minimal model program

For a rational adjoint divisor, a minimal model preserves a sufficiently divisible Veronese of the canonical ring: on a common resolution the two pullbacks differ by an effective divisor exceptional over the minimal model, which does not change sections. The leading asymptotic growth of those sections, and therefore the volume, is unchanged. This is why volume records the adjoint birational class rather than the choice of minimal model. For transcendental classes the analogous statement uses the invariance of positive volume under modifications and removal of the divisorial negative part, rather than a section ring.

8 Jiao's proof of upper semi-continuity of volume function

Theorem 8.1. *Let $X \longrightarrow T$ be a projective flat morphism of varieties and D a Cartier divisor on X. Suppose X_η is the generic fiber of f.*
Write $D_\eta := D|_{X_\eta}$ and $D_t := D|_{X_t}$ for every closed point $t \in T$, then

$$\operatorname{vol}\left(D_\eta\right) = \inf_{t \in T'} \operatorname{vol}\left(D_t\right)$$

for any dense subset $T' \subset T$.

Idea of proof. We divide the proof into several steps.
Step 1. We first reduce the problem to showing that when D_η is big, for any $\epsilon > 0$ there exists some $t \in T'$ such that

$$\operatorname{vol}(D_t) < \operatorname{vol}(D_\eta) + \epsilon.$$

Step 2. We then try to apply the Fujita approximation on the generic fiber; however, the problem is that the generic fiber is not defined over an algebraically closed field (which is needed for the Fujita approximation). Thus we first pass to the algebraic closure of the function field $\bar{K}$, and we get $g_{\bar{\eta}}^* D_{\bar{\eta}} \sim_{\mathbb{Q}} A_{\bar{\eta}} + E_{\bar{\eta}}$, where $Y_{\bar{\eta}}$ is smooth, $E_{\bar{\eta}} \geq 0$, $A_{\bar{\eta}}$ is ample, and

$$\operatorname{vol}\left(D_{\bar{\eta}}\right) \leq \operatorname{vol}\left(A_{\bar{\eta}}\right) + \epsilon,$$

and then descend to the generic fiber X_η. We then extend the Fujita approximation to the smooth compactification Y, as illustrated by the following diagram:

$$\begin{array}{ccccccc}
Y & \hookleftarrow & Y_{\bar{\eta}} & \longrightarrow & X_{\bar{\eta}} & \longrightarrow & X_\eta \longrightarrow X \\
\downarrow & & & \searrow & \downarrow & & \downarrow \quad \downarrow \\
T & \hookleftarrow & & & \operatorname{Spec}(\bar{K}) & \longrightarrow & \operatorname{Spec}(\eta) \longrightarrow T
\end{array}$$

where

$$g^*D \sim_{\mathbb{Q}} A + E,$$

with A ample over T. If

$$g_t^*D_t \sim_{\mathbb{Q}} A_t + E_t$$

is the Fujita approximation on the fiber, then we are done, since

$$\mathrm{vol}(D_\eta) \geq A_\eta^d = \mathrm{vol}(A_\eta) \geq \mathrm{vol}(D_\eta) - \epsilon$$

and

$$\mathrm{vol}(D_t) \geq A_t^d = \mathrm{vol}(A_t) \geq \mathrm{vol}(D_t) - \epsilon.$$

Thus we are done, since $A_t^d = A_\eta^d$ by the constancy of intersection numbers in flat families.

Step 3. In general, we need to take a Fujita approximation of $g_t^*D_t$, and get that for any $\epsilon' \in \mathbb{R}_+$ there exists a projective birational morphism $h_t : W_t \to Y_t$ such that

$$h_t^* g_t^* D_t \sim_{\mathbb{Q}} h_t^* (A_t + E_t) \sim_{\mathbb{Q}} A_t' + E_t'.$$

We then apply the asymptotic orthogonality estimate and the Hodge-type inequality:

$$A_\eta^{d-1} \cdot (A_\eta + E_\eta) \leq A_\eta^d + C'\sqrt{\epsilon}$$

and

$$\left(A_t^d\right)^{\frac{d-1}{d}} \left(A_t'^d\right)^{\frac{1}{d}} \leq \left(h_t^* A_t\right)^{d-1} \cdot A_t'$$

to show that $A_t'^d$ is not far from A_t^d:

$$A_t'^d \leq \left(\left(A_t^d\right)^{\frac{1}{d}} + \frac{C'\sqrt{\epsilon}}{\left(A_t^d\right)^{\frac{d-1}{d}}} \right)^d.$$

Thus $g_t^*D_t \sim_{\mathbb{Q}} A_t + E_t$ already serves as a good substitute for the Fujita approximation.

References

[Bou02] S. Boucksom, *On the volume of a line bundle*, International Journal of Mathematics **13** (2002), no. 10, 1043–1063.

[FKL16] M. Fulger, J. Kollár, and Brian Lehmann, *Volume and Hilbert function of $\mathbb{R}$-divisors*, Michigan Math. J. **65** (2016), no. 2, 371–387.

[HMX18] Christopher D. Hacon, James McKernan, and Chenyang Xu, *Boundedness of moduli of varieties of general type*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 4, 865–901.

[Jia25] Junpeng Jiao, *The volume function is upper semicontinuous on families of divisors*, 2025.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

Boucksom's Divisorial Zariski Decomposition

Kähler MMP Yi Li

Contents

1 Minimal multiplicities 1

2 The positive part 2

3 Exceptionality and the number of negative components 3

4 Surfaces and birational models 4

5 Research directions 4

Big picture

Divisorial Zariski decomposition removes the divisorial obstruction to nefness. Its positive part is modified nef; it need not be nef. We develop the current construction, prove the positivity assertion with the limiting step included, and distinguish this decomposition from the classical surface decomposition and from a semiample model.

1 Minimal multiplicities

For an effective divisor, the first question is which components are unavoidable in every representative of its class. Linear systems measure this by asymptotic vanishing orders. For a transcendental class there are no sections, but a positive current still has a generic multiplicity along each divisor. We therefore minimize these multiplicities. Allowing a small negative lower bound makes the construction stable as the class approaches the boundary of the pseudoeffective cone.

Let X be a compact Kähler manifold and fix a Kähler form ω. For a pseudoeffective class α, choose a current $T_\varepsilon \in \alpha$ with minimal singularities among currents bounded below by $-\varepsilon\omega$. Such currents exist by the potential-envelope construction in the currents note. Define

$$\nu(\alpha, D) = \lim_{\varepsilon \downarrow 0} \nu(T_\varepsilon, D), \qquad N(\alpha) = \sum_D \nu(\alpha, D) D, \qquad Z(\alpha) = \alpha - \{N(\alpha)\}.$$

The first limit is increasing: decreasing ε shrinks the admissible family, hence can only increase its minimal singularity type and generic Lelong numbers. Minimality is understood up to bounded potentials, so the definition does not depend on the chosen envelopes.

Proposition 1.1. *For every positive current $T \in \alpha$ and prime divisor D, $0 \le \nu(\alpha, D) \le \nu(T, D)$. In particular, if E is an effective real divisor, $N(\{E\}) \le E$.*

Proof. The current T belongs to every admissible family. Thus T_ε is less singular than T. Lelong-number comparison gives $\nu(T_\varepsilon, D) \le \nu(T, D)$, and passing to the increasing limit proves the assertion. Apply this to $T = [E]$ for the final assertion. ∎

2 The positive part

We use two analytic inputs from [Bou04]: the negative part is supported on finitely many prime divisors (Theorem 3.12), and a class is modified nef precisely when it admits, for every $\delta > 0$, a current bounded below by $-\delta\omega$ with zero generic Lelong number along every divisor. The finiteness theorem includes the stronger assertion that the support is an exceptional family with at most $\rho(X)$ members. Its proof is a remaining expansion in this draft.

Theorem 2.1 (Positivity of the Zariski projection)**.** *For every pseudoeffective class α, $Z(\alpha)$ is modified nef. Moreover $N(\alpha) = 0$ if and only if α is modified nef.*

Proof. The natural attempt is to remove the divisorial part of a least singular current. Its residual current has no divisorial Lelong numbers, which is exactly the condition we want. The difficulty is that the residual classes vary with the approximation parameter. We first remove the divisors and then correct this small cohomological error by smooth forms.

Apply Siu decomposition to the positive current $T_\varepsilon + \varepsilon\omega$ and write

$$T_\varepsilon = R_\varepsilon + [N_\varepsilon], \qquad N_\varepsilon = \sum_D \nu(T_\varepsilon, D) D.$$

Then $R_\varepsilon \ge -\varepsilon\omega$ and every generic divisorial Lelong number of R_ε is zero. By definition, $0 \le N_\varepsilon \le N(\alpha)$. Consequently all these divisors are supported on the same finite set $D_1, \dots, D_r$, and their coefficients converge to those of $N(\alpha)$.

Choose smooth forms η_i representing $\{D_i\}$, and put $\eta_\varepsilon = \sum_i (\nu(\alpha, D_i) - \nu(T_\varepsilon, D_i))\eta_i$. Their coefficients converge to zero, so $-c_\varepsilon\omega \le \eta_\varepsilon \le c_\varepsilon\omega$ with $c_\varepsilon \to 0$. The current $S_\varepsilon = R_\varepsilon - \eta_\varepsilon$ lies in the fixed class $Z(\alpha)$, has zero generic divisorial Lelong numbers, and satisfies $S_\varepsilon \ge -(\varepsilon + c_\varepsilon)\omega$. The stated criterion proves that $Z(\alpha)$ is modified nef. This smooth correction is necessary: the uncorrected residual currents are in varying classes $\alpha - \{N_\varepsilon\}$.

If $N(\alpha) = 0$, the first assertion applies directly to α. Conversely, the modified-nef criterion gives an admissible current with zero generic multiplicity for each lower bound $-\varepsilon\omega$.

Minimality forces $\nu(T_\varepsilon, D) = 0$ for every D and every ε, so $N(\alpha) = 0$. ∎

Proposition 2.2. *For $c > 0$, $N(c\alpha) = cN(\alpha)$ and $Z(c\alpha) = cZ(\alpha)$. For pseudoeffective α, β, $N(\alpha + \beta) \leq N(\alpha) + N(\beta)$ coefficientwise.*

Proof. Scaling currents identifies the admissible family for $c\alpha$ at lower bound $-\varepsilon\omega$ with the scaled family for α at lower bound $-(\varepsilon/c)\omega$. This proves homogeneity. The sum of two minimal currents at lower bound $-\varepsilon\omega$ is admissible for $\alpha + \beta$ at lower bound $-2\varepsilon\omega$. Minimality and additivity of generic Lelong numbers imply the asserted coefficient inequality after passing to the limit. ∎

3 Exceptionality and the number of negative components

Why should a negative part have only finitely many components? Positivity alone only bounds their total mass. The stronger input is that their classes cannot satisfy a linear relation: such a relation would let one move some of the supposedly fixed contribution to other divisors.

Definition 3.1. A finite family of prime divisors is exceptional if the cone generated by their classes meets the modified nef cone only at zero.

Proposition 3.2. *If an effective real divisor E has exceptional support, then $N(\{E\}) = E$. The classes of an exceptional family are linearly independent. Consequently such a family has at most $\rho(X)$ members.*

Proof. By Proposition 1.1, $E - N(\{E\})$ is effective and supported on the same exceptional family. Its class is $Z(\{E\})$, which is modified nef by Theorem 2.1. Exceptionality forces this class to vanish. A nonzero effective divisor has strictly positive intersection with ω^{n-1}, so an effective divisor of zero class is zero. Hence $E = N(\{E\})$.
Suppose $\sum_i a_i\{D_i\} = 0$. Separate positive and negative coefficients to obtain effective divisors E_+ and E_- with disjoint support and the same class. Their supports are exceptional subfamilies. The first assertion gives $E_+ = N(\{E_+\}) = N(\{E_-\}) = E_-$. Disjointness implies both are zero, so every $a_i = 0$. All these independent classes belong to $NS(X)_\mathbb{R}$, whose dimension is $\rho(X)$. ∎

Corollary 3.3. *If E is effective with exceptional support, $[E]$ is the only positive closed current in $\{E\}$.*

Proof. For any positive $T \in \{E\}$, Propositions 1.1 and 3.2 gives $\nu(T, D) \geq \mathrm{coeff}_D(E)$ on every component of E. Siu decomposition therefore gives $T - [E] \geq 0$. This positive current has zero cohomology class, and hence zero mass against ω^{n-1}. A positive current

of zero mass is zero. ∎

This proves the linear-independence and rigidity consequences in full. It does not yet prove that the support of $N(\alpha)$ is exceptional: that is the substantive input from Boucksom's theorem still used above. In particular, using the finiteness of $N(\alpha)$ in Theorem 2.1 and then using that theorem here must not be presented as an independent proof of finiteness.

4 Surfaces and birational models

On a smooth projective surface, a classical Zariski decomposition is $D = P + N$, with P nef, $N = \sum a_i C_i$ effective, the matrix $(C_i \cdot C_j)$ negative definite, and $P \cdot C_i = 0$ for all i. These stronger assertions must not be imported into higher dimensions.

Example 4.1. Let $\pi : X \to \mathbb{P}^2$ be the blowup of one point, H the pullback of a line and E the exceptional curve. For $a, b > 0$, the divisor $D = aH + bE$ has decomposition $P = aH$, $N = bE$. Indeed H is nef, $E^2 = -1$ and $H \cdot E = 0$. For integers $a, b \geq 0$, every effective divisor linearly equivalent to mD contains E with multiplicity at least mb: if its multiplicity is k and its remaining part is G, then $-mb = (kE + G) \cdot E = -k + G \cdot E \geq -k$. Thus multiplication by the canonical section of mbE identifies $H^0(X, maH)$ with $H^0(X, mD)$.

The birational transformation rule used to define the negative part on normal spaces is $N(\alpha) = \mu_* N(\mu^* \alpha)$ for modifications of smooth compact Kähler manifolds [Bou04, Lemma 5.8].

Proposition 4.2. *Let α be a pseudoeffective Bott–Chern class on the normal space, with pullbacks defined by local potentials. Assuming this transformation rule, $f_* N(f^* \alpha)$ is independent of the chosen Kähler resolution $f : Y \to X$ of a normal compact Kähler space.*

Proof. For resolutions $f : Y \to X$ and $g : Z \to X$, take a common Kähler resolution W with maps $p : W \to Y$ and $q : W \to Z$. Since $p^* f^* \alpha = q^* g^* \alpha$, the transformation rule gives $f_* N(f^* \alpha) = (fp)_* N((fp)^* \alpha) = (gq)_* N((gq)^* \alpha) = g_* N(g^* \alpha)$. ∎

The result downstairs is a Weil real divisor. Subtracting its class in Bott–Chern cohomology requires an additional Cartier condition; $\mathbb{Q}$-factoriality is a sufficient one. Modified nefness downstairs is a separate theorem, not a consequence merely of independence of the resolution; see [DHY23].

5 Research directions

The first issue is the behavior at the boundary of the pseudoeffective cone: homogeneity and coefficientwise subadditivity do not prove continuity there. A useful test is a surface with

infinitely many negative curves, comparing the negative parts along classes approaching a boundary ray. A second question is which additional hypotheses promote modified nefness to nefness on a chosen model. A third is the relation between a semiample positive part and finite generation: the blowup example explains the fixed-divisor mechanism, but modified nefness alone supplies neither sections nor a morphism. These are directions for investigation; no claim of novelty or complete literature status is made here.

References

[Bou04] Sébastien Boucksom, *Divisorial zariski decompositions on compact complex manifolds*, Annales scientifiques de l'Ecole normale supérieure, vol. 37, 2004, pp. 45–76.

[DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.

Generalized Kähler Pairs

Kähler MMP

Yi Li

Contents

1 Birational nef data 1

2 Why the higher nef part changes singularities 2

3 The Kähler formulation 3

4 What the framework provides 3

5 Research directions 4

Big picture

A generalized pair keeps a nef contribution on the birational model where it is actually nef. Its trace on the original space can lose positivity, and the discrepancy of this descent contributes to singularities. We establish this mechanism explicitly before using the language for transcendental classes.

1 Birational nef data

The canonical bundle formula suggests keeping two contributions on the base: a boundary describing singular fibres and a moduli contribution whose positivity reflects variation of the fibres. The second contribution may become nef only after a birational modification. Replacing it by an effective boundary would require sections and would change its singularity behavior. The definition instead retains the model on which it is nef.

In the divisor setting, a generalized pair consists of a normal variety X, a boundary $B \geq 0$, and birational nef data: a projective birational morphism $f : X' \to X$ and a nef real Cartier divisor M' on X'. Set $M_X = f_*M'$ and require $K_X + B + M_X$ to be real Cartier. On every higher model the nef part is the pullback of M'; these traces constitute the associated Cartier b-divisor $\boldsymbol{M}$. The datum includes the higher-model divisor, not just its trace M_X.

On a log resolution where the nef data descend, define B' by

$$K_{X'} + B' + M' = f^*(K_X + B + M_X).$$

The generalized log discrepancy of a prime divisor E on this model is $a(E, X, B + \boldsymbol{M}) = 1 - \mathrm{coeff}_E(B')$. The pair is generalized lc if all these discrepancies are nonnegative, and generalized klt if they are positive. Higher-model boundaries need not be effective.

Proposition 1.1. *Generalized discrepancies are independent of the model on which they are computed. If the nef data descend to X, they equal the ordinary discrepancies of (X, B).*

Proof. Let $g : Y \to X'$ be a further log resolution and put $M_Y = g^*M'$. Pulling back the defining equality and cancelling M_Y gives $K_Y + B_Y = g^*(K_{X'} + B')$. Thus higher discrepancies are the usual discrepancies of the subpair (X', B'). Comparing two models on a common resolution proves independence. If $M' = f^*M_X$ from the outset, the defining equality reduces to $K_{X'} + B' = f^*(K_X + B)$, as asserted. ∎

2 Why the higher nef part changes singularities

Theorem 2.1. *Suppose $K_X + B$ is real Cartier. In the divisor setting, generalized klt (respectively generalized lc) implies that (X, B) is klt (respectively lc).*

Proof. To compare the two notions of singularity, we must compare their crepant boundaries on the same resolution. Their difference is precisely the error between pulling back the nef trace and retaining the original nef divisor. The negativity lemma determines the sign of that error.
The Cartier assumption on both adjoint sums makes M_X real Cartier. On a log resolution carrying M', write $E = f^*M_X - M'$. This divisor is exceptional since its pushforward is zero, and $-E = M' - f^*M_X$ is f-nef. The negativity lemma therefore gives $E \geq 0$. If $\widetilde{B}$ is the ordinary boundary defined by $K_{X'} + \widetilde{B} = f^*(K_X + B)$, then $B' = \widetilde{B} + E$. Hence $\widetilde{B} \leq B'$. On any further resolution the same negativity argument applies, since the pulled-back nef part is still relatively nef. Thus every ordinary log discrepancy is at least the corresponding generalized discrepancy. Strict positivity or nonnegativity follows, respectively. ∎

The negativity lemma is a supporting birational theorem here. This proof does not assert the dlt version, whose formulation also involves the geometry of the non-klt locus.

Example 2.2. Let $f : X' \to \mathbb{P}^2 = X$ be the blowup of a point, with exceptional curve E and $H = f^*\mathcal{O}_{\mathbb{P}^2}(1)$. The divisor $H - E$ is basepoint free, hence nef. For $t \geq 0$ take $M' = t(H - E)$ and $B = 0$. Its trace is $M_X = t\mathcal{O}_{\mathbb{P}^2}(1)$. Since $K_{X'} = f^*K_X + E$, the defining equality gives $B' = (t - 1)E$. The generalized log discrepancy of E is consequently $2 - t$. For $0 \leq t < 2$ the resulting smooth subpair on X' is sub-klt, so the generalized pair is klt; at $t = 2$ it is lc but not klt; for $t > 2$ it is not lc. The underlying pair $(\mathbb{P}^2, 0)$ stays smooth throughout. Keeping only the trace M_X, which is ample for $t > 0$, would lose

this information.

3 The Kähler formulation

For the analytic formulation we follow [DHY23, Definition 2.7]. Let $\nu : X' \to X$ be a resolution of a normal compact Kähler space. The data comprise an SNC real divisor B' with $\nu_* B' \geq 0$ and a positive closed b-$(1,1)$-current $\boldsymbol{\beta}$ descending to X', whose class there is nef. Require

$$[K_{X'} + B' + \boldsymbol{\beta}_{X'}] = \nu^* \gamma, \qquad \gamma \in H^{1,1}_{\mathrm{BC}}(X).$$

Traces on a common higher model are defined by pullback of the descended current and then proper pushforward to other models. Pullback here is through local potentials; it is not a pullback operation on arbitrary distribution-valued forms. The relative version uses relative nefness over the specified base.

Generalized discrepancies are read from B' and its higher-model crepant boundaries. The formulation requires the supplied birational current data; a nef numerical class on X alone is not this datum. We use the independence and analytic negativity results of [DHY23] to pass between models in this formulation. The divisor proof above explains the sign, but does not substitute for these analytic results.

Proposition 3.1. *Adding the Cartier closure of a smooth closed $(1,1)$-form on X to the nef data leaves the generalized discrepancies unchanged, provided the new data still satisfy the required positivity and nefness conditions. In particular one can add a Kähler form.*

Proof. Add $\nu^*\omega$ to the nef trace upstairs and $[\omega]$ to γ downstairs. These terms cancel in the crepant identity, leaving B' unchanged. They cancel on every higher model as well. The coefficients defining discrepancies are therefore unchanged. ∎

4 What the framework provides

The canonical bundle formula motivates the definition: it creates a discriminant boundary and moduli data which may be nef only on a higher model. These are distinct contributions. The existence of a nef moduli class is an additional theorem with specific fibration hypotheses; it does not follow from the definition of generalized pairs.

Divisorial Zariski decomposition supplies another useful comparison. A modified-nef positive part on X is not automatically a nef b-class descending to a single model. To turn a decomposition into generalized pair data one must actually exhibit such a model and check the crepant identity. No finite generation or existence theorem is automatic from the decomposition alone.

The main subsequent results to be expanded are the analytic reduction from generalized klt to locally klt pairs, small $\mathbb{Q}$-factorial modifications, generalized adjunction, and the generalized Kähler MMP. Their proofs are not replaced by Theorem 2.1, which is explicitly the divisor argument. The linked vault proofs are the sources for these remaining expansions.

5 Research directions

One question is the distinction between a nef class and the finite positive combination of nef rational Cartier data used in NQC arguments. Before applying a scaling lemma, identify where this finite presentation is used. A second is descent: when can the trace of nef data be shown nef on the original space? The blowup example isolates the exceptional error whose sign controls discrepancies. A third is whether the modified-nef positive part of an adjoint class becomes nef on a model usable for generalized-pair constructions. Finally, canonical bundle formulas require separate control of the discriminant and the moduli part; positivity on curves alone does not establish Kähler nefness in higher dimension. These questions describe points of investigation, not a verified list of new open problems.

References

[DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.

Rational Curves on Compact Kähler Manifolds

Kähler MMP Yi Li

Contents

1 Pseudoeffectivity and covering curves 1

2 What Ou's argument supplies 2

3 Why negative canonical degree creates deformations 2

4 Passing through canonical singularities 3

5 Research directions 3

Big picture

The MMP needs rational curves for two reasons: they detect the negative canonical directions, and bounded anticanonical degree makes those directions controllable. The first task is global uniruledness; the second is the cone theorem. This note separates them and explains the bridge from positivity to deformation of curves.

1 Pseudoeffectivity and covering curves

Theorem 1.1 (BDPP and Ou)**.** *A smooth compact Kähler manifold X is uniruled if and only if K_X is not pseudoeffective [BDPP13, Ou25].*

For a projective manifold, BDPP proves the duality between pseudoeffective divisor classes and the closed cone of movable curve classes. The elementary inclusion is worth isolating.

Lemma 1.2. *An effective rational Cartier divisor has nonnegative intersection with a general member of a covering family of curves with irreducible general member. Consequently a pseudoeffective divisor class pairs nonnegatively with every class in the closed cone generated by such curves.*

Proof. A general member is not contained in the divisor. After normalization, the canonical section of a Cartier multiple restricts to a nonzero section on the curve. Its

divisor of zeros has nonnegative degree. Divide by the Cartier index. Finite positive combinations preserve the inequality, and continuity of the numerical pairing gives it for limits of effective divisor classes and limits of movable classes. ∎

The reverse inclusion is deep: it uses approximation of big classes, orthogonality of the fixed and moving contributions, and intersection inequalities. In a nonprojective Kähler space there may be very few algebraic curves, so this projective numerical argument cannot simply be repeated with the same curve cone.

2 What Ou's argument supplies

Ou proves Theorem 1.1 without assuming projectivity. The analytic ingredients include algebraicity criteria for foliations, positivity of relative adjoint classes, and a relative Albanese reduction. A foliation first detects directions of failed positivity; an algebraicity theorem then turns the leaves into compact algebraic geometry where rational connectedness can be established. The relative positivity argument controls the remaining base directions. These are the supporting results in [Ou25]; the separate BDPP supplement records their role rather than reproducing the full paper.

The theorem detects covering rational curves. It does not state that every negative curve is rational, or that all negative extremal rays have a uniform length bound. Those are further deformation and cone theorems.

3 Why negative canonical degree creates deformations

Lemma 3.1. *Let $f : C \to X$ be a map from a smooth projective curve of genus g to a smooth complex n-fold. The local dimension of the space of maps at f is at least $-K_X \cdot f_*C + n(1-g)$. If r distinct marked points have their images fixed, the analogous lower bound is $-K_X \cdot f_*C + n(1-g) - nr$.*

Proof. The tangent space to the deformation problem is $H^0(C, f^*T_X)$ and an obstruction space is $H^1(C, f^*T_X)$. Thus the dimension is at least their difference. Riemann–Roch gives $\chi(C, f^*T_X) = \deg(f^*T_X) + n(1-g)$, with $\deg(f^*T_X) = -K_X \cdot f_*C$. Fixing the images of marked points replaces f^*T_X by its twist by the negative marked divisor, reducing the Euler characteristic by nr. ∎

This calculation explains the sign in bend-and-break: more negative canonical degree forces more deformations. Producing a rational component still requires a compactification of the moving family and a degeneration argument. In the Kähler setting the necessary cycle compactness and control of singularities are part of the theorem, not consequences of Riemann–Roch alone. The threefold construction is developed in [HP16]; the cone chapter gives its MMP use.

4 Passing through canonical singularities

Proposition 4.1. *Let X be a normal projective variety with canonical singularities and rational Cartier K_X. Then K_X is pseudoeffective if and only if $K_{\widetilde{X}}$ is pseudoeffective on a projective resolution. Consequently X is uniruled if and only if K_X is not pseudoeffective.*

Proof. For a projective resolution $\mu : \widetilde{X} \to X$, write $K_{\widetilde{X}} = \mu^* K_X + E$ with $E \geq 0$ exceptional. Pullback preserves pseudoeffectivity, so adding E proves one implication. Conversely suppose $K_{\widetilde{X}}$ is pseudoeffective. Choose an ample Cartier divisor H on X. For every positive rational ε, the divisor $K_{\widetilde{X}} + \varepsilon \mu^* H$ is big, hence rationally linearly equivalent to an effective rational divisor. Push that divisor and the rational equivalence down to X. One obtains an effective representative of $K_X + \varepsilon H$. Letting ε tend to zero proves pseudoeffectivity of K_X. Uniruledness is invariant under bimeromorphic maps, so apply Theorem 1.1 to the smooth resolution. ∎

For a singular nonprojective Kähler target the corresponding argument requires analytic descent of currents with local potentials; pushing currents forward alone does not prove that assertion. Effectivity of E is exactly where canonical singularities enter. The same argument cannot be quoted unchanged for an arbitrary klt pair, where some discrepancies are negative.

5 Research directions

The useful interfaces are quantitative: how does analytic failure of positivity yield a curve with controlled degree, and how does that control survive singularities or a family? For generalized pairs, the nef data change the adjoint class without necessarily coming from a line bundle. A proof must identify whether it needs an actual curve deformation, a movable class, or a positive current. These objects play related but distinct roles in the program.

References

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.

[Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

The Cone Theorem in the Kähler MMP

Yi Li

Contents

1 A brief review of the cone theorem for projective varieties **3**

2 The generalized Mori cone and the Mori cone **5**

3 Höring–Peternell's approach for Kähler 3-folds **6**

3.1 Mori bend and break for Kähler threefolds 7

3.2 The cone theorem for Kähler 3-folds 8

4 Das–Hacon's approach for Kähler threefold dlt pairs **10**

5 A relative cone theorem for families of Kähler threefolds **12**

5.1 Relative setup 12

6 Hacon–Păun's approach in arbitrary dimension **16**

7 Applications of the cone theorem **18**

7.1 The contraction theorem and the elementary contractions 19

7.2 One step of the Kähler MMP 19

7.3 Families and higher dimension 19

8 Further discussions **20**

Big picture

The cone theorem locates the closed cone of curves' negative directions and shows they are cut out by rational curves of bounded degree — it is the entry point to the minimal model program, since it names exactly the extremal rays one is allowed to contract. In the Kähler category honest curves can be scarce, so the classical Mori cone $\overline{\mathrm{NE}}(X)$ is replaced by the **generalized Mori cone** $\overline{\mathrm{NA}}(X)$ of positive currents (Section 2). Section 1 reviews the projective cone theorem as a baseline; Sections 3

and 4 prove it for Kähler threefolds, first for terminal X (Höring–Peternell) and then for dlt pairs (Das–Hacon); Section 5 upgrades this to families of threefolds, the input needed for the 4-dimensional Kähler MMP with scaling; and Section 6 removes the dimension restriction entirely, at the cost of assuming BDPP — now a theorem of Ou. Section 7 collects the applications: the contraction theorem, one step of the Kähler MMP, and what each ingredient buys in families and in arbitrary dimension.

The major references are [HP24], [HP16], [DHYn23] and [DH24a].

The first main result is the cone theorem for Kähler threefolds.

Theorem 0.1 ([HP16, Theorem 6.3], [DH24a, Theorem 1.3])**.** *Let (X, B) be a compact Kähler threefold dlt pair. Assume further that $K_X + B$ is pseudo-effective. Then there exists a countable collection of rational curves $\{C_i\}_{i\in I}$ such that $0 < -(K_X + B) \cdot C_i \leq 6$ for all $i \in I$ and*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X+B)\geq 0} + \sum_{i\in I} \mathbb{R}_{\geq 0} \cdot [C_i]$$

Moreover, the following holds:

(1) For any Kähler class ω, there are only finitely many extremal rays $R_i := \mathbb{R}_{\geq 0} \cdot [C_i]$ such that $(K_X + B + \omega) \cdot R_i < 0$.

(2) For any $(K_X + B)$-negative extremal ray $R = \mathbb{R}_{\geq 0} \cdot [C_i]$, there is a nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ such that $\alpha^{\perp} \cap \overline{\mathrm{NA}}(X) = R$ and $\alpha = K_X + B + \eta$ for some Kähler class η.

The second main result is Hacon–Păun's proof of the cone theorem in arbitrary dimension.

Theorem 0.2 ([HP24, Theorem 0.5])**.** *Assume that the BDPP conjecture holds in dimension n (resp. in dimension $n-1$) — which is true by [Ou25, Theorem 1]. Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B + \boldsymbol{\beta})$ is generalized klt.*
Then there are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0} + \sum_{i\in I} \mathbb{R}^+ [\Gamma_i]$$

where $0 < -\left(K_X + B + \boldsymbol{\beta}_X\right) \cdot \Gamma_i \leq 2n$. Moreover, if $B + \boldsymbol{\beta}_X$ (or $K_X + B + \boldsymbol{\beta}_X$) is big, then I is finite.

Remark 0.3 (Kähler versus projective)**.** Two features distinguish these statements from the projective cone theorem (Theorem 1.1). First, the decomposition is of the *generalized* Mori cone $\overline{\mathrm{NA}}(X)$, not the Mori cone $\overline{\mathrm{NE}}(X)$: in the Kähler setting $\overline{\mathrm{NE}}(X)$ can be strictly smaller than $\overline{\mathrm{NA}}(X)$, and may even be trivial while X carries no curves at all (Remark 2.2), so it does not see enough of the geometry to admit a contraction

theory. Second, the pseudo-effectivity of $K_X + B$ — absent from the projective statement — is essential here: it is what provides a Zariski decomposition (in Section 3) or a well-behaved null locus (in Section 6) to run bend-and-break. The length bound itself is unchanged: $0 < -(K_X + B) \cdot \Gamma_i \leq 2n$ is the one-sided form of Kawamata's $-2n \leq (K_X + B) \cdot C < 0$.

⚠ **Warning.** The pseudo-effectivity of $K_X + B$ in Theorem 0.1 and Theorem 0.2 is not a technical convenience — it cannot be dropped: it is exactly what supplies the Zariski decomposition used to run bend-and-break in dimension three (Section 3), and the well-behaved null locus used in arbitrary dimension (Section 6). Without it there is no reason for the cone to be generated by countably many bounded-length rays at all.

1 A brief review of the cone theorem for projective varieties

Let us first give a brief review of the standard cone theorem that we are familiar with.

Theorem 1.1 ([Hac20, Theorem 6.4]; cf. [Bir12, Theorem 7.1]). *Let $(X/Z, B)$ be a projective klt pair of dimension d.*

(1) There are countably many rational curves $C_i \subset X$ such that

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+B\geq 0} + \sum_i R_i,$$

where the negative extremal rays R_i are all spanned by the curve class $[C_i]$.

(2) Each R_i is generated by the class of some rational curve C_i satisfying

$$-2d \leq (K_X + B) \cdot C_i < 0.$$

(3) If $F \subset \overline{\mathrm{NE}}(X)$ is a $(K_X + B)$-negative extremal face, then there exists a unique contraction morphism

$$\mathrm{cont}_F \colon X \to Z$$

such that a curve $C \subset X$ is contracted to a point if and only if $[C] \in F$.

(4) Let L be a Cartier divisor on X such that $L \cdot R = 0$ for some $(K_X + B)$-negative extremal ray R. Then there is a Cartier divisor L_Y on Y such that

$$L \sim (\mathrm{cont}_R)^* L_Y.$$

We refer to (2) as **Kawamata's length of extremal rays**. Note that here B is allowed to be an $\mathbb{R}$-divisor.

Idea of proof. Five steps: extract the length of extremal rays from Mori's bend-and-break; perturb B to a $\mathbb{Q}$-divisor so the rationality theorem applies; attach a contraction to the rational supporting hyperplane of each candidate face; show a Hahn–Banach separation argument forces these faces to generate the whole cone; and finally use the rationality theorem's denominator bound to show only finitely many rays are negative against any fixed ample perturbation.

Proof. We argue by induction on $\rho(X/S) = \dim N_1(X/S)$; the relative setting is essential throughout.

Step 1 (Length of extremal rays). For every $(K_X + B)$-negative extremal ray R there is a rational curve C with $[C] \in R$ and $-2d \le (K_X + B) \cdot C < 0$. This is Kawamata's bound, obtained from Mori's bend-and-break [Bir12, Cor. 2.8.4, Thm. 7.1]; it proves (2) and provides the boundedness used in Step 5.

Step 2 (Reduction to a $\mathbb{Q}$-divisor). A priori $K_X + B$ is only an $\mathbb{R}$-divisor. Fix a relatively ample divisor A and $\epsilon \in \mathbb{Q}_{>0}$. Write $K_X + B = \sum_{i=1}^{t} r_i D_i$ with D_i Cartier, and approximate the r_i by rationals r_i' so that $\sum_i (r_i - r_i')D_i + \frac{\epsilon}{3}A$ is relatively ample. Set $B' = \sum_i r_i' D_i - K_X$ and write $B' = (B')^+ - (B')^-$ with $(B')^{\pm}$ effective $\mathbb{Q}$-divisors without common component. For $r_i' - r_i$ small, the coefficients of $(B')^-$ are small, so there is an effective $\mathbb{Q}$-divisor B'' with small coefficients and $\frac{\epsilon}{3}A - (B')^- \sim_{\mathbb{Q}} B''$; shrinking further, $(X, (B')^+ + B'')$ is again klt. For every curve C,

$$C \cdot \left(K_X + (B')^+ + B'' + \tfrac{\epsilon}{3}A\right) = C \cdot (K_X + B + \epsilon A) - C \cdot \left(B - B' + \tfrac{1}{3}\epsilon A\right) \le C \cdot (K_X + B + \epsilon A),$$

so $\overline{\mathrm{NE}}(X/S)_{K_X + (B')^+ + B'' + \frac{\epsilon}{3}A \ge 0} \subset \overline{\mathrm{NE}}(X/S)_{K_X + B + \epsilon A \ge 0}$, and the assertion for (X, B) follows once it is known for the klt $\mathbb{Q}$-pair $(X, (B')^+ + B'')$. Hence we may assume B is a $\mathbb{Q}$-divisor, as required to apply the rationality theorem.

Step 3 (Contraction attached to a supporting plane). If $\rho(X/S) = 1$ there is nothing to prove, so assume $\rho(X/S) > 1$ and that $K_X + B$ is not relatively nef. For any relatively ample $\mathbb{Q}$-divisor H, the rationality theorem yields a rational nef threshold

$$r_H = \max\{t \in \mathbb{R} \mid H + t(K_X + B) \text{ is relatively nef}\} \in \mathbb{Q}, \qquad L_H := H + r_H(K_X + B).$$

Then L_H is relatively nef but not relatively ample, so $F_H := \overline{\mathrm{NE}}(X/S)_{L_H = 0}$ is a face: if $\alpha_1 + \alpha_2 \in F_H$ then, by nefness and the duality $\mathrm{Nef} = \overline{\mathrm{NE}}^*$ (Proposition 2.3), $\alpha_i \cdot L_H \ge 0$ forces $\alpha_1, \alpha_2 \in F_H$. Being a $(K_X + B)$-negative extremal face with a rational supporting function, F_H admits a contraction $h_H \colon X \to Y_H$.

Step 4 (The cone is generated by these faces). Let $\mathscr{C}$ be the smallest closed convex cone containing $\overline{\mathrm{NE}}(X/S)_{K_X + B \ge 0}$ and all F_H with $L_H \not\equiv_S 0$, i.e. $\mathscr{C} = \overline{\overline{\mathrm{NE}}(X/S)_{K_X + B \ge 0} + \sum_{L_H \not\equiv_S 0} F_H}$. We claim $\mathscr{C} = \overline{\mathrm{NE}}(X/S)$. If not, by Hahn–Banach there is a $\mathbb{Q}$-divisor M with $M \cdot v > 0$ on $\mathscr{C} \setminus \{0\}$ and $M \cdot v_0 < 0$ for some $v_0 \in \overline{\mathrm{NE}}(X/S)$. Then $M \not\equiv_S t(K_X + B)$: for if $M \equiv_S t(K_X + B)$ then $t < 0$ (both M and $-(K_X + B)$ are positive on some $F_H \setminus \{0\}$), while $M \cdot v_0 < 0$ forces $v_0 \in \overline{\mathrm{NE}}(X/S)_{K_X + B \ge 0} \setminus \{0\} \subset \mathscr{C} \setminus \{0\}$, contradicting $M > 0$ there. Now the dual

of $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}$ is $\overline{\mathrm{Amp}(X/S)+\mathbb{R}_{\geq 0}(K_X+B)}$ (dual of a half-space is a ray; dual of the Mori cone is the nef cone, Proposition 2.3), and M, being positive on $\overline{\mathrm{NE}}(X/S)_{K_X+B\geq 0}\setminus\{0\}$, is interior in it; hence $M = H + t(K_X + B)$ with H relatively ample and $t \in \mathbb{Q}_{>0}$. As M is not relatively nef, $t > r_H$. But $L_H \not\equiv_S 0$, so $F_H \subset \mathscr{C}$ and $M > 0$ on $F_H \setminus \{0\}$; taking C with $[C] \in F_H$,

$$C \cdot \big(H + t(K_X + B)\big) = \underbrace{C \cdot \big(H + r_H(K_X + B)\big)}_{=0} + (t - r_H)\, C \cdot (K_X + B) > 0$$

forces $C \cdot (K_X + B) > 0$, impossible on a negative face. This proves $\mathscr{C} = \overline{\mathrm{NE}}(X/S)$, which is (1). Part (3) is the contraction of Step 3, and part (4) is the induced descent of L via the base-point-free theorem [Bir12, Thm. 7.1].

Step 5 (Discreteness). Only finitely many R_i are $(K_X + B + \epsilon A)$-negative. For each such ray take the contraction $h_i\colon X \to Y_i$ and the generator $v_i \in R_i$ with $a(K_X + B) \cdot v_i = -1$. Complete $a(K_X + B)$ to a basis by relatively ample Cartier $H_1, \dots, H_{\rho-1}$; since $\dim N_1(X/Y_i) = 1$, the rationality theorem bounds the denominators $q_{ij} \leq a(b + 1)$ in $r_{ij}/a = p_{ij}/q_{ij}$, so $(a(b + 1))!\,(H_j \cdot v_i) \in \mathbb{Z}$. For $N \gg 0$ with $NA - H_j$ relatively ample and $(K_X + B + \epsilon A) \cdot v_i < 0$, one has $H_j \cdot v_i < NA \cdot v_i < N/(a\epsilon)$, so $H_j \cdot v_i$ takes finitely many values, whence finitely many rays. (Equivalently, by Step 1 the curves C_i with $(K_X + B + \epsilon A) \cdot C_i < 0$ satisfy $A \cdot C_i < 2b/\epsilon$, forming a bounded family with finitely many classes.) Letting $\epsilon \to 0$ gives countability of all the R_i. ∎

2 The generalized Mori cone and the Mori cone

Before entering the Kähler setting we fix the cones that replace the classical Mori cone. Throughout, X is a normal compact Kähler variety and classes live in $N_1(X) \subset H^{1,1}_{\mathrm{BC}}(X)^\vee$.

Definition 2.1 ([HP16, p. 226], [DH20, Def. 2.2])**.** We define the generalized Mori cone $\overline{\mathrm{NA}}(X) \subset N_1(X)$ to be the closed cone generated by the classes of positive closed $(1,1)$-currents (via the duality pairing between $H^{1,1}_{\mathrm{BC}}(X)$ and $H_{1,1}(X,\mathbb{R})$).
We define the **Mori cone**

$$\overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NA}}(X)$$

as the closure of the cone generated by the currents of integration T_C, where $C \subset X$ is an irreducible curve.

Intuition. Think of $\overline{\mathrm{NA}}(X)$ as the closed convex hull of every way a positive current can be swept out on X — the right receptacle for "effective classes" once honest curves are scarce, as happens in the Kähler category.

The cone $\overline{\mathrm{NA}}(X)$ serves as a substitute for the classical Mori cone in the Kähler setting.

For an $\mathbb{R}$-Cartier class D we write, as usual, $\overline{\mathrm{NA}}(X)_{D\geq 0} = \overline{\mathrm{NA}}(X) \cap \{\gamma : D \cdot \gamma \geq 0\}$, and similarly $\overline{\mathrm{NE}}(X)_{D\geq 0}$.

Remark 2.2. In the complex-analytic setting it may happen that

$$\overline{\mathrm{NE}}(X) \subsetneq \overline{\mathrm{NA}}(X).$$

For instance, there exist K3 surfaces with Picard number 0, carrying no curve C at all, yet many positive closed currents.
The inclusion can be strict even for projective manifolds. For projective K3 surfaces, $\rho(X) \leq 20 = h^{1,1}(X)$ by the Torelli theorem, so $\overline{\mathrm{NE}}(X)$ lies in the $\rho(X)$-dimensional subspace $N_1(X) \subsetneq H_{1,1}(X, \mathbb{R})$, while $\overline{\mathrm{NA}}(X)$ can contain classes outside $N_1(X)$, witnessing the strict inclusion.

The two facts about these cones that drive the Kähler cone theorem are a duality statement and a Kählerness (Kleiman-type) criterion.

Proposition 2.3 (Duality; [HP16, Prop. 3.9], [DP04, Cor. 0.4], [DH24b, Lem. 2.3])**.** *Let X be a normal compact Kähler variety with rational singularities. Then* $\mathrm{Nef}(X)$ *and* $\overline{\mathrm{NA}}(X)$ *are dual to each other via the canonical isomorphism $N^1(X) \to N_1(X)^*$. In particular, a class $\omega \in H^{1,1}_{\mathrm{BC}}(X)$ is Kähler if and only if $\omega \cdot \gamma > 0$ for all $\gamma \in \overline{\mathrm{NA}}(X) \setminus \{0\}$.*

Proposition 2.4 (Kählerness criterion for threefolds; [HP16, Cor. 3.16, Thm. 3.17–3.18], [CHP16, Thm. 3.7])**.** *Let X be a normal compact Kähler 3-fold with rational singularities such that every irreducible curve $C \subset X$ has $[C] \neq 0$ in $N_1(X)$, and let $[\eta] \in N^1(X)$. Then $[\eta]$ is a Kähler class if and only if $[\eta]$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$.*

We shall use Propositions 2.3 and 2.4 freely. We also record the Demailly–Păun description of the generators of $\overline{\mathrm{NA}}$ on a smooth model, which is the engine that upgrades the Mori cone statement to the generalized one.

Fact 2.5 (Demailly–Păun decomposition; [DP04, Cor. 0.4])**.** Let $\hat{X}$ be a compact Kähler manifold. Then $\overline{\mathrm{NA}}(\hat{X})$ is the closure of the convex cone generated by the classes $[\hat{\omega}]^2$, $[\hat{\omega}] \cdot [\hat{S}]$ and $[\hat{C}]$, where $\hat{\omega}$ is a Kähler form, $\hat{S}$ an irreducible surface and $\hat{C}$ an irreducible curve on $\hat{X}$.

3 Höring–Peternell's approach for Kähler 3-folds

Throughout this section X is a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities and K_X pseudo-effective. Let $K_X = P + N$ be the Zariski

decomposition with negative part $N = \sum a_j S_j$, and set

$$b := \max\{1,\ -K_X \cdot Z \mid Z \text{ a curve with } Z \subset S_{j,\mathrm{sing}} \text{ or } Z \subset S_j \cap S_{j'}\}, \qquad d := \max\{3, b\}.$$

3.1 Mori bend and break for Kähler threefolds

The heart of the argument is that curves of large anti-canonical degree break.

Theorem 3.1 (Mori bend and break; [HP16, Cor. 5.7, Lem. 5.8]). *With X and d as above:*

(a) (Breaking.) *If $C \subset X$ is a curve with $-K_X \cdot C > d$, then $[C] = [C_1] + [C_2]$ for non-zero effective (integral) 1-cycles C_1, C_2 on X.*

(b) (Rational curves.) *Let $\mathbb{R}^+[\Gamma_i]$ be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$ with Γ_i not very rigid. Then there is a curve C with $[C] \in \mathbb{R}^+[\Gamma_i]$ and $\dim_C \mathrm{Chow}(X) > 0$, and there is a rational curve C with $[C] \in \mathbb{R}^+[\Gamma_i]$.*

Idea of proof. Large anti-canonical degree forces the curve to deform; its deformations sweep out a uniruled surface from the Zariski decomposition of K_X, and ordinary surface bend-and-break on that uniruled surface produces the splitting, or a rational curve, after pushing forward.

Proof. *(a) Breaking.* Since $-K_X \cdot C > d \geq b$, the curve C is deformable: $\dim_C \mathrm{Chow}(X) > 0$ (a curve of sufficiently large anti-canonical degree on a Kähler 3-fold moves, [HP16, §5]). Let $(C_t)_{t \in T}$ be a family of deformations. By [HP16, Lem. 5.4] there is a unique Zariski surface S_j containing all C_t, with

$$K_{S_j} \cdot C < K_X \cdot C < -d.$$

By definition of b we have $C \not\subset S_{j,\mathrm{sing}}$ (otherwise $-K_X \cdot C \leq b \leq d$, contradicting $-K_X \cdot C > d$). Let $\pi_j \colon \hat{S}_j \to S_j$ be the composition of normalization and minimal resolution; the strict transform $\hat{C}$ of C is well defined and

$$K_{\hat{S}_j} \cdot \hat{C} \leq K_{S_j} \cdot C < -3.$$

Since K_{S_j} is not pseudo-effective, $\hat{S}_j$ is a smooth uniruled surface, so Mori bend-and-break on surfaces gives an effective 1-cycle $\sum_{k=1}^{m} C_k$ with $m \geq 2$, $[\sum_k C_k] = [\hat{C}]$, and $K_{\hat{S}_j} \cdot C_1 < 0$, $K_{\hat{S}_j} \cdot C_2 < 0$. As $K_{\hat{S}_j}$ is π_j-nef (minimal resolution), any C_k with $(\pi_j)_* C_k = 0$ would be π_j-vertical, hence $K_{\hat{S}_j} \cdot C_k \geq 0$; thus $(\pi_j)_* C_1 \neq 0$ and $(\pi_j)_* C_2 \neq 0$. Therefore

$$[C] = (\pi_j)_*[\hat{C}] = \sum_{k=1}^{m} (\pi_j)_*[C_k],$$

with the first two summands non-zero, so $[C]$ splits as $[C_1'] + [C_2']$.

(b) Rational curves in an extremal ray. Let $R = \mathbb{R}^+[\Gamma_i]$ be extremal with Γ_i not very rigid, and set

$$m := \min\{k \in \mathbb{N} \mid \exists\, C \subset X,\ [C] \in R,\ \dim_{kC} \mathrm{Chow}(X) > 0\}.$$

We show $m = 1$. If $m > 1$, choose C realizing m with deformation family $(C_t)_{t\in T}$; by minimality of m the family has no fixed component, so a general C_t has every irreducible component C_t' with $\dim_{C_t'} \mathrm{Chow}(X) > 0$, and extremality gives $[C_t'] \in R$ — contradicting minimality. Hence there is C with $[C] \in R$ and $\dim_C \mathrm{Chow}(X) > 0$, which is the first assertion of (b). For the rational curve, by [HP16, Lem. 5.4] the deformations of C lie in some Zariski surface S_j with $K_{S_j} \cdot C < K_X \cdot C < 0$; after a general deformation $C \not\subset S_{j,\mathrm{sing}}$, and on $\hat{S}_j$ the strict transform satisfies $K_{\hat{S}_j} \cdot \hat{C} \le K_{S_j} \cdot C < 0$. Surface bend-and-break yields an effective $\mathbb{Q}^+$-cycle $\sum_k \alpha_k C_k$ with $[\sum_k \alpha_k C_k] = [\hat{C}]$ and C_1 a rational curve with $K_{\hat{S}_j} \cdot C_1 < 0$; as $K_{\hat{S}_j}$ is π_j-nef, $(\pi_j)_* C_1 \neq 0$, so

$$[C] = \sum_k \alpha_k (\pi_j)_* [C_k], \qquad \alpha_1 (\pi_j)_* [C_1] \neq 0.$$

By extremality, $[C_1]$ — the class of a rational curve — is a positive multiple of $[\Gamma_i]$, so R contains the class of a rational curve. ∎

Remark 3.2. In the language of [CHP16, Def. 4.5, Prop. 4.6], the theorem says that $K_X + \Delta$ has the **bend-and-break property**: there is $d \in \mathbb{Q}^+$ such that every curve C with $-(K_X + \Delta) \cdot C > d$ splits as $[C] = [C_1] + [C_2]$. This is what makes Kawamata's length effective.

3.2 The cone theorem for Kähler 3-folds

Theorem 3.3 ([HP16, Theorem 6.3])**.** *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudo-effective. Then there exist $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i\in I}$ of curves on X such that*

$$0 < -K_X \cdot \Gamma_i \le d \qquad \text{and} \qquad \overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X \ge 0} + \sum_{i\in I} \mathbb{R}^+ [\Gamma_i]\,.$$

Moreover, if the ray $\mathbb{R}^+ [\Gamma_i]$ is extremal in $\overline{\mathrm{NA}}(X)$, there exists a rational curve C_i on X with $[C_i] \in \mathbb{R}^+ [\Gamma_i]$.

The proof splits into a statement for the Mori cone and an upgrade to the generalized Mori cone. We first record the tool that gives closedness.

Proposition 3.4 (Closedness under bounded length; [HP16, Lem. 6.1]). *Let $N \subset \overline{\mathrm{NA}}(X) \subset N_1(X)$ be a closed convex cone, and suppose there are $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i\in I}$ with $0 < -K_X \cdot \Gamma_i \le d$ and*

$$N = \overline{N_{K_X\ge 0} + \textstyle\sum_{i\in I} \mathbb{R}^+[\Gamma_i]}.$$

Then $N = N_{K_X\ge 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i]$. The same holds with a dlt or generalized log canonical class in place of K_X.

Proof. Set $V := N_{K_X\ge 0} + \sum_i \mathbb{R}^+[\Gamma_i]$; we must show $\overline{V} = V$. Since $V \subset \overline{\mathrm{NA}}(X)$, V contains no line and is the convex hull of its extremal rays, so it suffices to show that every extremal ray $\mathbb{R}^+[r]$ of $\overline{V}$ with $K_X \cdot r < 0$ lies in V. Fix a Kähler form ω and $\varepsilon > 0$ with $(K_X + \varepsilon\omega)\cdot r < 0$. Then

$$N = \overline{N_{K_X+\varepsilon\omega\ge 0} + \textstyle\sum_{j\in J} \mathbb{R}^+[\Gamma_j]},$$

the sum over the $j \in I$ with $(K_X + \varepsilon\omega)\cdot\Gamma_j < 0$; for these, $-K_X\cdot\Gamma_j \le d$ gives $\omega\cdot\Gamma_j \le d/\varepsilon$, so by boundedness of the ω-degree there are only finitely many classes, say $J = \{1,\dots,q\}$, and $N_{K_X+\varepsilon\omega\ge 0} + \sum_{j=1}^{q} \mathbb{R}^+[\Gamma_j]$ is already closed.

Write $r = \lim_m (r_m + s_m)$ with $(K_X+\varepsilon\omega)\cdot r_m \ge 0$ and $s_m = \sum_{j=1}^{q} \lambda_{j,m}\Gamma_j$. As $\omega\cdot(r_m+s_m) \to \omega\cdot r$ and $\omega \ge 0$ on N, the sequences $\omega\cdot r_m$ and $\omega\cdot\lambda_{j,m}\Gamma_j$ are bounded, so after passing to a subsequence $r_m \to r_\infty$ and $\lambda_{j,m}\Gamma_j \to \mu_j\Gamma_j$. Then $r = r_\infty + \sum_j \mu_j\Gamma_j$ with all summands in $\overline{V}$; since r is extremal in $\overline{V}$, each is a non-negative multiple of r. But $(K_X + \varepsilon\omega)\cdot r_\infty \ge 0$ while $(K_X + \varepsilon\omega)\cdot r < 0$ forces $r_\infty = 0$, so some $\mu_{j_0}\Gamma_{j_0}$ is a positive multiple of r; hence $\mathbb{R}^+[r] = \mathbb{R}^+[\Gamma_{j_0}] \subset V$. ∎

Proposition 3.5 (Weak cone theorem, Mori cone version; [HP16, Theorem 6.2]). *With X as in Theorem 3.3, there are $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i\in I}$ with $0 < -K_X\cdot\Gamma_i \le d$ and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\ge 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i],$$

and every extremal ray of $\overline{\mathrm{NE}}(X)$ contains the class of a rational curve.

Proof. Take $d = \max\{3, b\}$ as above. There are at most countably many curve classes in $\overline{\mathrm{NE}}(X)$ (integral curves are lattice points), and we pick a representative Γ_i of each class with $0 < -K_X\cdot\Gamma_i \le d$. Set

$$V := \overline{\mathrm{NE}}(X)_{K_X\ge 0} + \sum_{0<-K_X\cdot\Gamma_i\le d} \mathbb{R}^+[\Gamma_i],$$

which is closed by Proposition 3.4. Fix a Kähler form ω with $\omega\cdot C \ge 1$ for every irreducible curve C (possible after rescaling, since only finitely many classes have ω-degree < 1); the same then holds for effective 1-cycles.

By Proposition 3.4 it suffices to show $\overline{\mathrm{NE}}(X) = \overline{V}$, i.e. $[C] \in V$ for every irreducible curve C. We induct on $\ell := \omega\cdot C$. The case $\ell = 0$ is trivial. Assume the claim for all curves

of degree $\leq \ell - 1$, and let $\ell - 1 < \omega \cdot C \leq \ell$. If $-K_X \cdot C \leq d$ we are done. Otherwise Theorem 3.1(a) gives $[C] = [C_1] + [C_2]$ with C_i effective 1-cycles; since $\omega \cdot C_i \geq 1$ we get $\omega \cdot C_i \leq \ell - 1$, so by induction $[C_1], [C_2] \in V$ and hence $[C] \in V$.
Finally, if $\mathbb{R}^+[\Gamma_i]$ is extremal, then Theorem 3.1(b) (together with the fact that a very rigid K_X-negative curve is already a rational curve of bounded degree) produces a rational curve in the ray. ∎

Proof of Theorem 3.3. Set $V := \overline{\mathrm{NA}}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$ with the Γ_i from Proposition 3.5. By Proposition 3.4 it suffices to prove $\overline{\mathrm{NA}}(X) \subset \overline{V}$.
Let $\pi \colon \hat{X} \to X$ be a desingularization and $\alpha \in \overline{\mathrm{NA}}(X)$. By the Kähler–Mori push-forward, $\alpha = \pi_*(\hat{\alpha})$ for some $\hat{\alpha} \in \overline{\mathrm{NA}}(\hat{X})$. By Fact 2.5, $\overline{\mathrm{NA}}(\hat{X})$ is generated by classes of the form $[\hat{\omega}]^2$, $[\hat{\omega}] \cdot [\hat{S}]$ and $[\hat{C}]$; we show the π-image of each type lies in $\overline{V}$.
Case $\hat{\alpha} = [\hat{\omega}]^2$. Since $\pi^* K_X$ is pseudo-effective, $K_X \cdot \alpha = \pi^* K_X \cdot \hat{\alpha} \geq 0$, so $\alpha \in \overline{\mathrm{NA}}(X)_{K_X \geq 0} \subset \overline{V}$.
Case $\hat{\alpha} = [\hat{C}]$. Put $C = \pi_* \hat{C}$, so $\alpha = [C]$. Under the inclusion $\overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}^+[\Gamma_i] \hookrightarrow \overline{\mathrm{NA}}(X)_{K_X \geq 0} + \sum \mathbb{R}^+[\Gamma_i]$, Proposition 3.5 gives $[C] \in V$.
Case $\hat{\alpha} = [\hat{\omega}] \cdot [\hat{S}]$. If $\pi^* K_X \cdot [\hat{\omega}] \cdot [\hat{S}] \geq 0$ then $\alpha \in \overline{\mathrm{NA}}(X)_{K_X \geq 0}$. Suppose instead $\pi^* K_X \cdot [\hat{\omega}] \cdot [\hat{S}] < 0$. We claim $\hat{\alpha} \in \overline{\mathrm{NE}}(\hat{X})$; since $\pi_* \overline{\mathrm{NE}}(\hat{X}) = \overline{\mathrm{NE}}(X)$, this reduces to the curve case. By the projection formula $\pi(\hat{S})$ is not a point, and as X has isolated singularities $\pi(\hat{S})$ is a surface S. The restriction $K_X|_S$ is not pseudo-effective, so S is one of the Zariski surfaces S_j; in particular S is uniruled and so is $\hat{S}$. Let $\pi_j \colon \hat{S}_j \to \hat{S} \subset \hat{X}$ be the composition of normalization and minimal resolution; then $\hat{S}_j$ is uniruled and projective, whence $H^2(\hat{S}_j, \mathcal{O}) = 0$ and the Chern class map $\mathrm{Pic}\, \hat{S}_j \to H^2(\hat{S}_j, \mathbb{Z})$ is surjective, so $\mathrm{NS}_{\mathbb{R}}(\hat{S}_j) = H^{1,1}(\hat{S}_j) \cap H^2(\hat{S}_j, \mathbb{R})$. Thus $[\pi_j^* \hat{\omega}]$ is a nef and big $\mathbb{R}$-divisor class, a limit of ample $\mathbb{Q}$-divisor classes H_m represented by effective $\mathbb{Q}$-1-cycles $[C_m]$. One checks $(\pi_j)_*[C_m] \to \hat{\alpha}$: by duality it suffices that for every real closed $(1,1)$-form η,

$$[\eta] \cdot (\pi_j)_*[C_m] = [\pi_j^* \eta] \cdot [C_m] \longrightarrow \int_{\hat{S}_j} \pi_j^* \eta \wedge \pi_j^* \omega = [\eta] \cdot [\hat{\omega}] \cdot [\hat{S}],$$

which follows from the construction of $[C_m]$. Hence $\hat{\alpha} \in \overline{\mathrm{NE}}(\hat{X})$ and $\alpha \in \overline{V}$. ∎

4 Das–Hacon's approach for Kähler threefold dlt pairs

The result of the previous section holds more generally for dlt pairs; we omit the terminal case and prove the dlt version, Theorem 0.1. This is [DH24a, Theorem 1.3] (see also [DH20, Thm. 2.22]).

Idea of proof. The strategy follows Section 3: Boucksom's nefness criterion [Bou04] reconciles metric and algebraic nefness of $K_X + B$; Kähler threefold dlt pairs satisfy Mori bend-and-break (Claim 4.1), giving Kawamata's length; and the closedness

Proposition 3.4 together with the duality Proposition 2.3 reduces the generalized cone to the Mori cone via the Demailly–Păun generators (Fact 2.5). The one genuinely new feature is the exposed-point argument, which replaces the induction on ω-degree used in Theorem 3.3.

Claim 4.1 (Bend and break for dlt pairs; [DH24a, Claim 4.8])**.** *There is $d > 0$ such that if $C \subset X$ is a curve with $-(K_X + B + \omega) \cdot C > d$, then $[C] = [C_1] + [C_2]$ with C_1, C_2 non-zero integral effective 1-cycles.*

The following lemma is pure cone geometry and encapsulates the closedness input; we omit its proof.

Lemma 4.2 (Cone geometry / closedness)**.** *Let $V := \overline{\mathrm{NA}}(X)_{(K_X+B)\geq 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i]$, where the Γ_i are rational curves with $0 < -(K_X + B) \cdot \Gamma_i \leq d$. Then V is a closed convex cone. In particular $\overline{\mathrm{NA}}(X) = V$ if and only if $\overline{\mathrm{NA}}(X) = \overline{V}$; equivalently, it suffices to show that every generator of $\overline{\mathrm{NA}}(X)$ lies in $\overline{V}$.*

Proof of Theorem 0.1. *Reduction to strongly $\mathbb{Q}$-factorial X.* By standard arguments (cf. [DHP24, Cor. 5.3]), passing to a strongly $\mathbb{Q}$-factorial model (which exists by [DH20, Lem. 2.28]) we may assume X is strongly $\mathbb{Q}$-factorial.
Extremal rays are spanned by short rational curves. Let K be a compact convex slice of $\overline{\mathrm{NA}}(X)$ of Euclidean dimension $n - 1$, $n = \dim_{\mathbb{R}} H^{1,1}_{\mathrm{BC}}(X)$ (so the cone on K is $\overline{\mathrm{NA}}(X)$), and let R be an extremal ray corresponding to an exposed point $\{x\} \in K$. By definition there is a class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ with $K \subset \alpha_{\geq 0}$ and $\{x\} = K \cap \alpha^{\perp}$, so $R = \overline{\mathrm{NA}}(X) \cap \alpha^{\perp}$. Since α is non-negative on $\overline{\mathrm{NA}}(X)$ it is nef (Proposition 2.3). If $(K_X + B) \cdot R < 0$, then for $0 < a \ll 1$ the class $\beta := \alpha - a(K_X + B)$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$, hence Kähler; writing $\alpha = a(K_X + B + \frac{1}{a}\beta)$ and rescaling we may assume $\alpha \equiv K_X + B + \omega$ with ω Kähler. By the existence of flipping/divisorial contractions ([DH24a, Thms. 6.2, 6.9]), $R = \mathbb{R}^+[\Gamma]$ for a curve Γ, and since $-(K_X + B)$ is relatively ample for the contraction with (X, B) dlt, Γ may be taken to be a rational curve with $0 > (K_X + B) \cdot \Gamma \geq -6$.
Countability. As X is Kähler, a Douady space argument shows there are at most countably many such extremal rays $\{\mathbb{R}^+[\Gamma_i]\}_{i\in I}$.
Reduction to $\overline{\mathrm{NA}}(X) = \overline{V}$. Put $V := \overline{\mathrm{NA}}(X)_{K_X+B\geq 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i]$. Following the proof of Theorem 3.3 it suffices to show $\overline{\mathrm{NA}}(X) = \overline{V}$; by Lemma 4.2 the point is the closedness of V. Suppose $\overline{\mathrm{NA}}(X) \setminus \overline{V} \neq \emptyset$. Since $\overline{V}$ is convex and $\overline{\mathrm{NA}}(X)$ is generated by K, there is an exposed point $x \in K$ not in $\overline{V}$; as V contains everything non-negative against $K_X + B$, the ray R through x is $(K_X + B)$-negative. By the previous paragraph there is a rational curve Γ with $0 < -(K_X + B) \cdot \Gamma \leq 6$ and $R = \mathbb{R}^+[\Gamma]$ — but every such ray lies in $\overline{V}$ by definition of V, a contradiction. Hence $\overline{\mathrm{NA}}(X) = \overline{V} = V$.
The finiteness statements (1)–(2) follow from the boundedness $0 < -(K_X + B) \cdot \Gamma_i \leq 6$

together with a modified-Kähler perturbation argument ([DH20, Rmk. 4.12–4.13]); see also Proposition 2.4 for the nef supporting class in (2). ∎

Remark 4.3 (Höring–Peternell versus Das–Hacon)**.** Theorem 0.1 generalizes Theorem 3.3 from the terminal absolute case ($B = 0$) to dlt pairs, and sharpens the conclusion with the finiteness (1) and the nef supporting class (2). Both proofs reduce to the closedness of V (Proposition 3.4), but they differ in how they show every generator of $\overline{\mathrm{NA}}(X)$ lies in $\overline{V}$. Höring–Peternell induct on the ω-degree of curves, applying bend-and-break *directly on* $\overline{\mathrm{NE}}(X)$, and only then transport the conclusion to $\overline{\mathrm{NA}}(X)$ through the three Demailly–Păun generator types $[\hat{\omega}]^2, [\hat{\omega}] \cdot [\hat{S}], [\hat{C}]$ (Fact 2.5). Das–Hacon instead argue *on* $\overline{\mathrm{NA}}(X)$ *directly*: they take a compact slice, single out an exposed point, and produce a nef supporting class $\alpha = K_X + B + \omega$ with ω Kähler. The latter route is better suited to pairs — it never leaves $\overline{\mathrm{NA}}(X)$ — and it is exactly the form that survives into arbitrary dimension in Section 6.

5 A relative cone theorem for families of Kähler threefolds

For the Kähler MMP with scaling in dimension four one needs a cone theorem in families of Kähler threefolds. We first fix the (routine but notation-heavy) relative apparatus, then state and completely prove the theorem, following [DHP24, §5, §7–8].

5.1 Relative setup

Throughout, $f : X \to T$ is a proper surjective morphism from a normal Kähler variety X to a smooth curve T, and $W \subset T$ is a fixed compact subset.

Definition 5.1 (Semi-stable klt pair; [DHP24, Def. 8.1])**.** Let $B \geq 0$ be an effective $\mathbb{Q}$-divisor on X. We say $(X, B/T; W)$ is a **semi-stable klt pair** if $(X, X_w + B)$ is plt for every $w \in W$. Equivalently:

(1) every fibre X_w, $w \in W$, is reduced, irreducible and normal;
(2) $\operatorname{Supp} B$ contains no fibre X_w (so $B_w := B|_{X_w}$ makes sense);
(3) $K_X + B$ is $\mathbb{Q}$-Cartier and (X_w, B_w) is klt for every $w \in W$.

Definition 5.2 (Relative N_1 and $\overline{\mathrm{NA}}$; [DHP24, Def. 8.2, p. 57])**.** Assume every fibre of f is reduced, irreducible and normal. For $u \in U$, an open neighbourhood of W in T, and τ a real closed bidimension-$(1,1)$ current on X_u, and η a real closed $(1,1)$-form on $X_U := f^{-1}U$ with local potentials, set $\tau(\eta) := (\iota_{u,*}\tau)(\eta) = \tau(\eta|_{X_u})$, where $\iota_u : X_u \hookrightarrow X$ is the inclusion. Let $N_1(X/T, W)$ be the real vector space generated by such currents τ on fibres X_w, $w \in W$, modulo $\tau_1 \equiv \tau_2$ iff $\tau_1(\alpha) = \tau_2(\alpha)$ for all $\alpha \in H^{1,1}_{\mathrm{BC}}(X_U)$ and

some neighbourhood $U \supset W$. Dually, $N^1(X/T, W)$ is the space of classes $\alpha \in H^{1,1}_{\mathrm{BC}}(X_U)$ modulo $\alpha_1 \equiv \alpha_2$ iff $[\tau](\alpha_1) = [\tau](\alpha_2)$ for every such τ on every X_w, $w \in W$ (a direct limit over shrinking $U \supset W$). We write $\overline{\mathrm{NA}}(X/T, W) \subset N_1(X/T, W)$ for the closed cone generated by classes of positive closed currents, and $[C] \in N_1(X/T, W)$ for the class of a curve $C \subset X_w$, $w \in W$.

A Kähler form ω on X restricts to a Kähler form $\omega_t := \omega|_{X_t}$ on every fibre; we call such ω **relatively Kähler**.

Theorem 5.3 (Relative cone theorem; [DHP24, Theorem 8.5]). *Let $f : X \to T$ be a proper surjective morphism from a Kähler 4-fold X to a curve T with $f_*\mathcal{O}_X = \mathcal{O}_T$, let $W \subset T$ be a compact subset, and let $(X, B/T; W)$ be a semi-stable klt pair. Fix a Kähler form ω on X. Then:*

(a) there are finitely many curve classes $\{C_i\}_{i\in J}$ over W with $0 > (K_X + B) \cdot C_i \geq -6$;

(b) for each $t \in W$,

$$\overline{\mathrm{NA}}(X_t) = \overline{\mathrm{NA}}(X_t)_{(K_{X_t}+B_t+\omega_t)\geq 0} + \sum_{i\in J} \mathbb{R}^{\geq 0}[C_i].$$

Suppose moreover that $K_{X_t} + B_t + \omega_t$ is nef for all $t \in W$, with $\omega_t := \omega|_{X_t}$ Kähler, and set $\lambda := \inf\{s \geq 0 \mid K_{X_t} + B_t + s\omega_t \text{ is nef for all } t \in W\}$. If $\lambda > 0$, then there are finitely many classes $\{C_i\}_{i\in I}$, $I \subset J$, over W such that:

(1) $C_i \subset X_t$ for some $t \in W$, and $\mathbb{R}^{\geq 0}[C_i]$ is a $(K_{X_t} + B_t)$-negative extremal ray of $\overline{\mathrm{NA}}(X_t)$ with $(K_{X_t} + B_t + \lambda\omega_t) \cdot C_i = 0$;

(2) if $C \subset X_t$ satisfies $(K_{X_t} + B_t + \lambda\omega_t) \cdot C = 0$ for some $t \in W$, then $[C] \equiv \sum_{i\in I} c_i[C_i]$ in $N_1(X/T, W)$ with $c_i \in \mathbb{R}^{\geq 0}$;

(3) if $\omega \in N^1(X/T, W)$ is general, then $|I| = 1$.

Idea of proof. The proof assembles three ingredients: the threefold cone theorem applied fibre by fibre, a boundedness lemma for families of curves of bounded degree, and an elementary genericity lemma in linear algebra.

Corollary 5.4 (Fibrewise finiteness; [DHP24, Cor. 5.3]). *In the setting of Theorem 5.3, for every $t \in W$ the pair (X_t, B_t) is a compact Kähler threefold dlt pair with $K_{X_t} + B_t$ pseudo-effective, and for the Kähler class ω_t there are only finitely many $(K_{X_t} + B_t + \omega_t)$-negative extremal rays of $\overline{\mathrm{NA}}(X_t)$; each is generated by a rational curve C_i with $0 > (K_{X_t} + B_t) \cdot C_i \geq -6$.*

Proof. By Definition 5.1, (X_t, B_t) is a klt — in particular dlt — compact Kähler threefold pair for every $t \in W$; that $K_{X_t} + B_t$ is pseudo-effective is part of the standing hypotheses of [DHP24, §5] on the family (guaranteed there by the semi-stability and properness of f over W, which make pseudo-effectivity of $K_{X_t} + B_t$ a closed, hence fibrewise constant,

condition on the compact set W). Granting this, the claim is exactly Theorem 0.1(1) applied to (X_t, B_t) with the Kähler class ω_t: finiteness of the $(K_{X_t} + B_t + \omega_t)$-negative extremal rays, together with Theorem 0.1's main assertion that each $(K_{X_t} + B_t)$-negative extremal ray of $\overline{\mathrm{NA}}(X_t)$ is generated by a rational curve C_i with $0 < -(K_{X_t}+B_t)\cdot C_i \le 6$. ∎

Lemma 5.5 (Boundedness in families; [DHP24, Lemma 8.4]). *Let $f : X \to T$, ω relatively Kähler, $W \subset T$ compact, and $M > 0$. Let $\{C_i\}_{i\in I}$ be the set of f-vertical curves with $f(C_i) \subset W$ and $\omega \cdot C_i \le M$. Then the C_i belong to finitely many families of curves; in particular they represent finitely many classes in $N_1(X/T, W)$.*

Proof. Fix a Kähler form η on X. For each $t \in W$ there is $\epsilon_t > 0$ with $(\omega - \epsilon_t f^*\eta)|_{X_t}$ Kähler on X_t, hence (openness of the Kähler condition) $(\omega - \epsilon_t f^*\eta)|_{X_s}$ Kähler for s near t. By compactness of W we may choose a single $\epsilon > 0$ such that $(\omega - \epsilon f^*\eta)|_{X_t}$ is Kähler for every t in a neighbourhood of W. Then for every i,

$$\eta \cdot C_i < \tfrac{1}{\epsilon}\,\omega \cdot C_i \le \tfrac{M}{\epsilon}.$$

Since the relative cycle space (Barlet/Douady space) over W has only finitely many irreducible components of η-degree bounded by M/ϵ, the curves C_i belong to finitely many families. ∎

Lemma 5.6 (General and very general Kähler class; [DHP24, Lemma 7.1]). *Let V be a (finite-dimensional) real vector space and $\mathcal{C} \subset V$ a cone not contained in any hyperplane. Let $\{C_i\}_{i\in I} \subset V^*$ be a finite (resp. countable) collection with $\omega \cdot C_i := C_i(\omega) > 0$ for all $\omega \in \mathcal{C}$, and $C_i \neq \lambda C_j$ for $i \neq j$ and $\lambda \in \mathbb{R}$. Fix $D \in V$. Then there is a finite (resp. countable) union of hyperplanes $\mathcal{H} \subset V$ such that for every $\omega \in \mathcal{C} \setminus \mathcal{H}$ and every $t \in \mathbb{R}$, $(D + t\omega) \cdot C_i = 0$ for at most one $i \in I$.*

Proof. Since $C_i \neq \lambda C_j$ for $i \neq j$, the annihilator $\langle C_i, C_j\rangle^\perp \subset V$ has codimension 2. For $i \neq j$ set

$$\mathcal{H}(C_i, C_j) := \{\omega \in \mathcal{C} \mid D + t\omega \in \langle C_i, C_j\rangle^\perp \text{ for some } t \in \mathbb{R}\}.$$

If $D + t\omega \in \langle C_i, C_j\rangle^\perp$ then $t\omega \in \langle C_i, C_j\rangle^\perp - D$, so ω lies in the linear span of $\langle C_i, C_j\rangle^\perp - D$, a subspace of codimension ≥ 1 (a hyperplane, since D may shift it off the origin but the span is still proper as $\langle C_i, C_j\rangle^\perp \neq V$); hence $\mathcal{H}(C_i, C_j)$ is contained in a hyperplane. Set $\mathcal{H} := \bigcup_{i\neq j} \mathcal{H}(C_i, C_j)$, a finite (resp. countable) union of hyperplanes. For $\omega \in \mathcal{C} \setminus \mathcal{H}$ and any $t \in \mathbb{R}$, $D + t\omega \notin \langle C_i, C_j\rangle^\perp$ for all $i \neq j$; in particular $(D + t\omega) \cdot C_i \neq 0$ for at most one $i \in I$. ∎

Proof of Theorem 5.3. *Part (a)–(b).* By Corollary 5.4, for each $t \in T$ there are finitely many $(K_{X_t} + B_t + \omega_t)$-negative extremal rays, generated by curves C_i, $i \in J_t$, with $0 > (K_{X_t} + B_t) \cdot C_i = (K_X + B) \cdot C_i \ge -6$ (note $(K_X + B) \cdot C_i = (K_{X_t} + B_t) \cdot C_i$ since $C_i \subset X_t$ and $K_X + B$ restricts to $K_{X_t} + B_t$ by the plt/adjunction hypothesis of Definition 5.1). Let $J := \bigcup_{t\in T} J_t$. Since $\omega \cdot C_i = \omega_t \cdot C_i < -(K_{X_t} + B_t) \cdot C_i \le 6$ for $i \in J_t$, Lemma 5.5 (with $M = 6$)

shows J is finite. This proves (a); statement (b) is Corollary 5.4 restated fibre by fibre, indexed over the finite set J in place of J_t (enlarging the generating set only adds classes already in $\overline{\mathrm{NA}}(X_t)_{(K_{X_t}+B_t+\omega_t)\geq 0}$ for $t' \neq t$, which are non-negative on $K_{X_t}+B_t+\omega_t$ and hence absorbed into the first summand on X_t).
Part (1). Suppose $K_{X_t}+B_t+\omega_t$ is nef for all $t \in W$ and $\lambda > 0$. Set

$$\Lambda := \{t \in W \mid K_{X_t}+B_t+\lambda\omega_t \text{ is nef but not Kähler}\} \subset T.$$

If $\Lambda = \emptyset$, then (arguing as in the proof of Lemma 5.5, using compactness of W to pass from fibrewise Kählerness to a uniform statement over a neighbourhood of W) $K_X+B+\lambda\omega$ would be relatively Kähler over a neighbourhood of W, contradicting the definition of λ as an infimum. So $\Lambda \neq \emptyset$. For $t \in \Lambda$, the class $K_{X_t}+B_t+\lambda\omega_t$ is nef but not Kähler, so by the Kählerness criterion (Proposition 2.4) the face

$$F_t := (K_{X_t}+B_t+\lambda\omega_t)^{\perp} \cap \overline{\mathrm{NA}}(X_t) \neq \{0\}.$$

By part (b) (equivalently Corollary 5.4), F_t is generated by finitely many of the classes $[C_i]$, $i \in J$, each spanning a $(K_{X_t}+B_t)$-negative extremal ray. Let

$$\Gamma := \left\{ C \subset X_t \;\middle|\; \begin{array}{l} t \in \Lambda, \quad (K_{X_t}+B_t+\lambda\omega_t)\cdot C = 0, \\ [C] \text{ spans a } (K_{X_t}+B_t)\text{-negative extremal ray} \end{array} \right\}.$$

For $C \in \Gamma$, since $(K_{X_t}+B_t+\lambda\omega_t)\cdot C = 0$,

$$\omega \cdot C = \omega_t \cdot C = -\tfrac{1}{\lambda}(K_{X_t}+B_t)\cdot C \leq \tfrac{6}{\lambda}.$$

By Lemma 5.5 (with $M = 6/\lambda$), the curves in Γ belong to finitely many families, hence finitely many numerical classes $\{C_i\}_{i\in I}$, $I \subset J$. This proves (1).
Part (2). Let $C \subset X_t$ satisfy $(K_{X_t}+B_t+\lambda\omega_t)\cdot C = 0$ for some $t \in W$; then $[C] \in F_t$. By part (b) and part (1), F_t is generated by the classes $\{[C_j]\}_{j\in J'}$ for some finite $J' \subset I$ (the ones lying in F_t), so $[C] = \sum_{j\in J'} c_j[C_j]$ with $c_j \in \mathbb{R}_{\geq 0}$ in $H^{1,1}_{\mathrm{BC}}(X_t)$, and hence also, by definition of the relative N_1-space (Definition 5.2), in $N_1(X/T, W)$.
Part (3). Apply Lemma 5.6 with $V = N^1(X/T, W)$, $\mathcal{C}$ the cone of relatively Kähler classes (which is not contained in a hyperplane, since it is open and non-empty), $D = K_X+B$, and $\{C_i\}_{i\in I}$ the finitely many curve classes of part (1) (pairwise non-proportional after discarding repetitions). The lemma produces a finite union of hyperplanes $\mathcal{H} \subset N^1(X/T, W)$ such that for $\omega \in \mathcal{C}\setminus\mathcal{H}$ — i.e. for ω general — and $t = \lambda$, $(K_X+B+\lambda\omega)\cdot C_i = 0$ for at most one $i \in I$. Since $\lambda > 0$ was defined so that this vanishing holds for at least one i (part (1)), it holds for exactly one, i.e. $|I| = 1$. ∎

Remark 5.7 (Comparison with the threefold and arbitrary-dimensional theorems)**.** Theorem 5.3 is not a new proof of a cone theorem in dimension four: it is the threefold theorem (Theorem 0.1), applied fibre by fibre to the Kähler threefolds X_t, $t \in W$,

packaged with a uniformity statement across the family. The genuinely new content is Lemma 5.5, which replaces the individual boundedness arguments of Sections 3 and 4 (bounded ω-degree on a single X) by a compactness argument over $W \subset T$: it is what lets the finitely-many-rays statement on each fibre assemble into a single finite set J working uniformly over W. Lemma 5.6 is unrelated to positivity; it is used only to force genericity of ω in part (3), and has no analogue in Sections 3 and 6 because those sections work with a single fixed variety rather than a family.

6 Hacon–Păun's approach in arbitrary dimension

Assuming BDPP — which now holds by [Ou25, Theorem 1] — Hacon and Păun prove the cone theorem in all dimensions, Theorem 0.2. The engine is the production of an α-trivial rational curve; the following is the local statement.

Proposition 6.1 ([HP24, Theorem 0.5, simpler version])**.** *Assume BDPP. Let (X, ω) be Kähler, $(X, B + \boldsymbol{\beta})$ generalized klt, and suppose $\alpha = K_X + B + \boldsymbol{\beta} + \omega$ is nef. Then there is an α-trivial rational curve C with*

$$0 > (K_X + B + \boldsymbol{\beta}) \cdot C = \omega \cdot C \geq -2 \dim X.$$

We use the following inputs, all in the generalized-Kähler category: Nakamaye's theorem, Kawamata's subadjunction and the canonical bundle formula for generalized pairs ([HP24]), and Cao–Păun's positivity of relative adjoint classes.

Idea of proof. Locate a maximal-dimensional component Z of the null locus of α, subadjunct to it to see that K_Z is not pseudo-effective, pass to its MRC fibration, and show the generic fibre F — now projective — carries a short α-trivial rational curve via the ordinary cone theorem.

Proof of Proposition 6.1. If α is Kähler there is nothing to prove, so assume α is nef but not Kähler; then $\mathrm{Null}(\alpha)$ has a positive-dimensional component.
Step 1 (null locus as a log canonical center). By Nakamaye's theorem, the restricted non-Kähler locus $E^{as}_{nK}(\alpha)$ coincides with $\mathrm{Null}(\alpha)$, and there is a Kähler current η with weak analytic singularities, $\{\eta\} \in \alpha$, whose Lelong upper-level set equals $E^{as}_{nK}(\alpha)$. Let Z be a maximal-dimensional irreducible component of $\mathrm{Null}(\alpha)$, and let c be the log canonical threshold of $(X, B + \boldsymbol{\beta}_X)$ with respect to η near a general point of Z. Choose a common log resolution $\nu\colon X' \to X$ with $K_{X'} + B_{X'} + \boldsymbol{\beta}_{X'} = \nu^*(K_X + B + \boldsymbol{\beta}_X)$, $\boldsymbol{\beta}_{X'}$ nef, and $\nu^*\eta = \eta' + F$ with $F \geq 0$ effective, $\eta' \geq 0$ a positive closed current, and $F + B_{X'}$ SNC. Then Z is a component of $\nu((B_{X'} + cF)^{=1})$ and not contained in $\nu((B_{X'} + cF)^{>1})$. Setting $\boldsymbol{\beta} + c\boldsymbol{\eta}$ with $\boldsymbol{\eta} = \overline{\eta'}$, the analytic subset Z is a generalized log canonical center of the generalized pair

$(X, B + c\nu_*F + \boldsymbol{\beta} + c\boldsymbol{\eta})$.
Step 2 (subadjunction). By Kawamata subadjunction for generalized Kähler pairs, on the normalization Z^ν of Z,

$$\big(K_X + B + c\nu_*F + \boldsymbol{\beta} + c\boldsymbol{\eta}\big)\big|_{Z^\nu} = K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu},$$

where $(Z^\nu, B_{Z^\nu} + \gamma_{Z^\nu})$ is a generalized pair, so $B_{Z^\nu} \geq 0$ and γ_{Z^ν} is pseudo-effective. Since $K_X + B + c\nu_*F + \boldsymbol{\beta} + c\boldsymbol{\eta} \equiv (c+1)\alpha - \omega$, restricting gives $K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu} \equiv (c+1)\alpha_{Z^\nu} - \omega_{Z^\nu}$, where $\alpha_{Z^\nu} = \alpha|_{Z^\nu}$, $\omega_{Z^\nu} = \omega|_{Z^\nu}$. With $k := \nu_{\mathrm{num}}(\alpha_{Z^\nu}) < \dim Z$ (so $\alpha_{Z^\nu}^k \not\equiv 0$, $\alpha_{Z^\nu}^{k+1} \equiv 0$),

$$\big(K_{Z^\nu} + B_{Z^\nu} + \gamma_{Z^\nu}\big) \cdot \alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z - k - 1} = -\,\alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z - k} < 0,$$

the strict inequality because α_{Z^ν} is nef and $\alpha_{Z^\nu}^k \not\equiv 0$.
Step 3 (MRC fibration). Since $B_{Z^\nu} \geq 0$ and γ_{Z^ν} is pseudo-effective, the displayed negativity forces K_{Z^ν} (hence $K_{Z'}$ for any resolution $Z' \to Z^\nu$) to be non-pseudo-effective. Consider the MRC fibration $Z' \to Y$, non-trivial as $K_{Z'}$ is not pseudo-effective; after passing to a higher model it is a morphism with general fibre F. As F is rationally connected, $h^2(\mathcal{O}_F) = 0$, so F is projective, and F is rationally connected.
Step 4 (α_F is not big). For any $\epsilon > 0$ and $t > 0$,

$$\big(K_{Z'} + B_{Z'} + \gamma_{Z'} + (1-\epsilon)\omega_{Z'} + t\alpha_{Z'}\big) \cdot \alpha_{Z'}^k \cdot \omega_{Z'}^{\dim Z - k - 1} = -\epsilon\, \alpha_{Z^\nu}^k \cdot \omega_{Z^\nu}^{\dim Z - k} < 0,$$

and since $\gamma_{Z'}$ is pseudo-effective and $B_{Z'}^{<0}$ is $(Z' \to Z^\nu)$-exceptional, $K_{Z'} + (1-\epsilon)\omega_{Z'} + t\alpha_{Z'}$ is not pseudo-effective for any $t > 0$. As Y is not uniruled, K_Y is pseudo-effective; applying Cao–Păun positivity of the relative adjoint class to $Z' \to Y$ shows $K_F + (1-\epsilon)\omega_F + t\alpha_F$ is not pseudo-effective for any $t > 0$. In particular α_F is not big (else it would be pseudo-effective for $t \gg 0$).
Step 5 (fibre-type contraction). By the projective cone theorem applied to $(F, (1-\epsilon)\omega_F)$, there are finitely many $(K_F + (1-\epsilon)\omega_F)$-negative extremal rays, and (comparing with α_F, which is nef) a non-empty finite subcollection of them is α_F-trivial. Let $\eta\colon F \to \bar{F}$ contract such a face; then $\alpha_F = \eta^*\alpha_{\bar{F}}$ with $\alpha_{\bar{F}}$ ample. If η were birational, α_F would be big — contradiction. Hence η is of fibre type, and F is covered by α_F-trivial rational curves C.
Step 6 (length). By bend-and-break we may take $0 < -K_F \cdot C \leq 2\dim F$, hence $(1-\epsilon)\omega_F \cdot C < 2\dim F \leq 2\dim X$. Thus Z' (and so Z) is covered by α-trivial rational curves C with $0 < \omega \cdot C < \frac{2}{1-\epsilon}\dim X$. These lie in finitely many numerical classes, so letting $\epsilon \to 0$ we may assume $0 < \omega \cdot C \leq 2\dim X$. Finally, by α-triviality,

$$0 < -(K_X + B + \boldsymbol{\beta}_X) \cdot C = \omega \cdot C \leq 2\dim X. \qquad \blacksquare$$

Idea of proof. By Proposition 6.1, every nef adjoint class $\alpha = \Theta + \omega$ that is not Kähler already carries a short α-trivial rational curve; feeding this into the exposed-point argument of Theorem 0.1 verbatim gives the cone theorem in dimension n.

Proof of Theorem 0.2. Write $\Theta := K_X + B + \boldsymbol{\beta}_X$. By Proposition 6.1, every nef adjoint class $\alpha = \Theta + \omega$ (ω Kähler) that is not Kähler carries an α-trivial rational curve C with $0 < -\Theta \cdot C = \omega \cdot C \leq 2n$. This is the Mori bend-and-break input; the rest of the argument is the exposed-point argument of Theorem 0.1.

Extremal rays are short rational curves. Let K be a compact convex slice of $\overline{\mathrm{NA}}(X)$ of Euclidean dimension $\dim_{\mathbb{R}} H^{1,1}_{\mathrm{BC}}(X) - 1$, and R the ray of an exposed point $\{x\} = K \cap \alpha^{\perp}$, where α is nef (Proposition 2.3) with $K \subset \alpha_{\geq 0}$. If $\Theta \cdot R < 0$, then for $0 < a \ll 1$ the class $\beta := \alpha - a\Theta$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$, hence Kähler; rescaling, $\alpha \equiv \Theta + \omega$ with ω Kähler, so the previous paragraph applies and $R = \mathbb{R}^+[\Gamma]$ for a rational curve Γ with $0 < -\Theta \cdot \Gamma \leq 2n$. A Douady space argument (as X is Kähler) shows there are at most countably many such rays $\{\mathbb{R}^+[\Gamma_i]\}_{i \in I}$.

Closedness and conclusion. Put $V := \overline{\mathrm{NA}}(X)_{\Theta \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$. By Proposition 3.4 (generalized-lc form, with $0 < -\Theta \cdot \Gamma_i \leq 2n$), V is closed, so it suffices to prove $\overline{\mathrm{NA}}(X) = \overline{V}$. If not, since $\overline{V}$ is convex and $\overline{\mathrm{NA}}(X)$ is generated by K, there is an exposed point $x \in K \setminus \overline{V}$; as $V \supset \overline{\mathrm{NA}}(X)_{\Theta \geq 0}$, its ray R is Θ-negative, hence $R = \mathbb{R}^+[\Gamma]$ for a rational curve with $0 < -\Theta \cdot \Gamma \leq 2n$ by the previous step. But every such ray lies in $\overline{V}$ by definition of V — a contradiction. Thus

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{\Theta \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i], \qquad 0 < -\Theta \cdot \Gamma_i \leq 2n.$$

Finally, if $B + \boldsymbol{\beta}_X$ (or Θ) is big, then after a modified-Kähler perturbation ω' every Θ-negative Γ_i has bounded ω'-degree while ω' is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$; hence the classes $[\Gamma_i]$ cannot accumulate, and I is finite. ∎

Remark 6.2 (What changes in higher dimension)**.** The arbitrary-dimensional argument (Proposition 6.1) replaces the surface-specific input of Section 3 — the Zariski decomposition of a pseudo-effective K_X on a threefold, and Mori bend-and-break on the resulting uniruled surfaces — by two dimension-free tools. Nakamaye's theorem describes the null locus of the nef-but-not-Kähler class α as the Lelong set of a Kähler current, and BDPP (now Ou's theorem [Ou25]) applied to the MRC fibration of a maximal null-locus component produces the covering rational curves. The subadjunction of Step 2 plays the role that the surface adjunction $K_{S_j} \cdot C < K_X \cdot C$ did in dimension three. This is precisely why the theorem is conditional *only* on BDPP: once BDPP holds in the relevant dimension, the bend-and-break input — a short α-trivial rational curve — is available in every dimension, and the exposed-point argument of Section 4 then finishes verbatim.

7 Applications of the cone theorem

The cone theorem is the first pillar of the (Kähler) minimal model program: it locates the $(K_X + B)$-negative extremal rays, which are exactly the rays one contracts. We record the

principal consequences.

7.1 The contraction theorem and the elementary contractions

Each negative extremal ray can be contracted in the Kähler category, and the contraction falls into one of three types.

Theorem 7.1 (Contraction theorem; [HP16, Thm. 1.3, Cor. 7.7–7.9, Thm. 7.12], [DH24a]). *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities and K_X pseudo-effective (or, more generally, (X, B) a $\mathbb{Q}$-factorial compact Kähler 3-fold dlt pair with $K_X + B$ pseudo-effective). Let $R = \mathbb{R}^+[\Gamma]$ be a $(K_X + B)$-negative extremal ray of $\overline{\mathrm{NA}}(X)$, with supporting nef class $\alpha = K_X + B + \eta$ as in Theorem 0.1(2). Then the contraction $\mathrm{cont}_R \colon X \to Y$ exists in the Kähler category, and exactly one of the following holds:*

- *(i)* (divisorial) cont_R *is bimeromorphic and contracts a prime divisor S — to a point if the nef dimension $n(\alpha) = 0$, to a curve if $n(\alpha) = 1$;*
- *(ii)* (flipping / small) cont_R *is bimeromorphic and small, contracting only finitely many curves; the R-flip $X \dashrightarrow X^+$ then exists;*
- *(iii)* (fibre type / Mori fibre space) $\dim Y < \dim X$ *and $-(K_X + B)$ is cont_R-ample, exhibiting $X \to Y$ as a Mori fibre space.*

The nef supporting class α of Theorem 0.1(2) is precisely the input for building cont_R: one shows α is semi-ample (a Kähler analogue of the base-point-free theorem, through a nef-and-big Kählerness criterion), and cont_R is the associated fibration. The fibre-type case realises X as uniruled; for Kähler 3-folds whose MRC fibration has surface base this is [HP15, Thm. 3.15].

7.2 One step of the Kähler MMP

Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler 3-fold dlt pair. If $K_X + B$ is nef, X is already a minimal model. Otherwise Theorem 0.1 produces a $(K_X + B)$-negative extremal ray R, and Theorem 7.1 contracts it: a divisorial contraction or a flip replaces (X, B) by a new $\mathbb{Q}$-factorial dlt pair with smaller Picard number or milder discrepancies, while a fibre-type contraction ends the program with a Mori fibre space. Iterating and invoking termination runs the Kähler MMP in dimension three [DH20, DH24a]. The finiteness in Theorem 0.1(1) is exactly what makes MMP *with scaling* well defined: for a fixed Kähler ω only finitely many rays are $(K_X + B + \omega)$-negative, so the nef threshold is attained and the scaling proceeds.

7.3 Families and higher dimension

The relative cone theorem (Theorem 5.3) plays the same role for the Kähler MMP *with scaling in families* and is a key ingredient in the 4-dimensional Kähler MMP [DHP24]. In

arbitrary dimension, Theorem 0.2 supplies the extremal rays for generalized klt pairs, so that — granting the existence of the corresponding contractions and flips — the same one-step picture is expected to run the generalized Kähler MMP.

Key takeaways

- The cone theorem always has the same shape: countably many bounded-length rational curves generate the negative part of the relevant cone, with the nef/pseudo-effective hypothesis on $K_X + B$ supplying either a Zariski decomposition or a well-behaved null locus to run bend-and-break.
- The generalized Mori cone $\overline{\mathrm{NA}}(X)$, not the classical $\overline{\mathrm{NE}}(X)$, is the right object in the Kähler category (Remark 2.2).
- Each negative extremal ray contracts to a divisorial, flipping, or fibre-type morphism (Theorem 7.1); iterating this is one step of the Kähler MMP, and the finiteness clause of the cone theorem is exactly what makes MMP with scaling well posed.
- Passing to families (Theorem 5.3) or arbitrary dimension (Theorem 0.2) reuses the threefold argument verbatim, once a boundedness lemma over the family, resp. BDPP, is granted.

8 Further discussions

(1) The optimal bound for Kawamata's length of extremal rays has recently been established; the sharp constant improves the $2n$ (resp. 6 in dimension three) above.

(2) Fujino generalised the cone theorem to the log canonical and semi-log canonical cases.

(3) Corti–Lazić gave an alternative approach to the cone theorem via finite generation.

References

[Bir12] Caucher Birkar, *Lectures on birational geometry*, 2012.

[Bou04] Sébastien Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76.

[CHP16] Frédéric Campana, Andreas Höring, and Thomas Peternell, *Abundance for Kähler threefolds*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 971–1025. MR 3552019

[DH20] Omprokash Das and Christopher Hacon, *The log minimal model program for kähler 3-folds*, 2020.

[DH24a] _____, *On the minimal model program for kähler 3-folds*, 2024.

[DH24b] ———, *Transcendental minimal model program for projective varieties*, 2024.

[DHP24] Omprokash Das, Christopher Hacon, and Mihai Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68. MR 4719824

[DHYn23] Omprokash Das, Christopher Hacon, and José Ignacio Yáñez, *MMP for generalized pairs on kähler 3-folds*, 2023.

[DP04] Jean-Pierre Demailly and Mihai Păun, *Numerical characterization of the kähler cone of a compact kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274.

[Hac20] Christopher Hacon, *Lectures on the minimal model program*, 2020, Lecture notes.

[HP15] Andreas Höring and Thomas Peternell, *Mori fibre spaces for Kähler threefolds*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 1, 219–246. MR 3329195

[HP16] ———, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264. MR 3437871

[HP24] Christopher Hacon and Mihai Păun, *On the canonical bundle formula and adjunction for generalized kähler pairs*, 2024.

[Ou25] Wenhao Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

Analytic Contractions and Their Extension

Kähler MMP Yi Li

Contents

1 From a negative locus to an analytic target 1

2 Kollár–Mori's extension theorem 2

3 Use in the minimal model program 4

Big picture

An extremal ray is numerical data; a contraction is a holomorphic map. The analytic task is to construct the target and its structure sheaf, then establish the positivity needed for the next MMP step. This base chapter explains the formal-neighborhood mechanism and gives the extension theorem used in families. Detailed contraction constructions belong to the threefold/fourfold MMP chapter.

1 From a negative locus to an analytic target

For an exceptional divisor, positivity of its conormal bundle controls the infinitesimal neighborhoods of the proposed contracted locus. Vanishing of their first higher direct images lets functions descend through successive thickenings. Analytic contraction theorems of Grauert and Fujiki make this formal information effective under their precise negativity and vanishing hypotheses. The classical surface instance says that a compact curve configuration on a smooth complex surface is exceptional when its intersection matrix is negative definite. These analytic existence theorems are supporting inputs here.

The sign is visible on a blowup of a smooth point: the exceptional divisor is $E \simeq \mathbb{P}^{n-1}$ and $\mathcal{O}_E(-E) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. The conormal bundle is positive precisely when the normal direction can be collapsed. In an MMP application, identifying the exceptional locus and proving the needed positivity precede the application of the contraction theorem. Kählerness of the target is another step, treated separately in the preservation chapter.

2 Kollár–Mori's extension theorem

The distinction between a divisor and a fibre is essential here. For a general Cartier divisor the conormal bundle need not descend under a contraction. For a fibre over a smooth curve it is trivial. This observation removes the apparent projection-formula obstruction in the original argument.

Theorem 2.1 (Extension of a contraction, [KM92, Proposition 11.4])**.** *Let $p : \mathcal{X} \to \Delta$ be a proper flat deformation over a smooth pointed disc, with central fibre X_0. Let $f_0 : X_0 \to Y_0$ be a proper surjective morphism of normal compact complex spaces. Suppose that $(f_0)_*\mathcal{O}_{X_0} = \mathcal{O}_{Y_0}$ and $R^1(f_0)_*\mathcal{O}_{X_0} = 0$. After shrinking Δ, there are a proper flat deformation $q : \mathcal{Y} \to \Delta$ of Y_0 and a proper morphism $F : \mathcal{X} \to \mathcal{Y}$ over Δ extending f_0, with $F_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$.*

Proof. We separate construction of the target, convergence, and the direct-image assertion. The analytic inputs are the existence of semi-universal deformations of compact complex spaces, the relative Douady space and its open graph locus, and analytic approximation [Gra72, Art68]. These inputs are used as theorems; the sheaf-theoretic steps are proved below.

The conormal bundle. Choose a coordinate t on Δ. Flatness makes multiplication by t injective on $\mathcal{O}_{\mathcal{X}}$, so $X_0 = (t = 0)$ is a principal Cartier divisor. In particular, for every $j \geq 1$,

$$\mathcal{I}^j/\mathcal{I}^{j+1} \simeq \mathcal{O}_{X_0}, \qquad \mathcal{I} = (t).$$

The isomorphism sends the class of $t^j a$ to $a|_{X_0}$. It follows that $R^1(f_0)_*(\mathcal{I}^j/\mathcal{I}^{j+1}) = 0$. No descent of an arbitrary line bundle is involved.

Infinitesimal targets and flatness. Put $A_j = \mathbb{C}[t]/(t^{j+1})$ and $X_j = (|X_0|, \mathcal{O}_{\mathcal{X}}/(t^{j+1}))$. On $|Y_0|$ define the sheaf of rings $\mathcal{B}_j = (f_0)_*\mathcal{O}_{X_j}$, where pushforward is along the underlying continuous map. The sequences of successive powers of t and the assumed vanishing give

$$0 \longrightarrow \mathcal{O}_{Y_0} \xrightarrow{t^j} \mathcal{B}_j \longrightarrow \mathcal{B}_{j-1} \longrightarrow 0.$$

Here the first arrow denotes the identification with the top graded piece, not multiplication on an independently chosen lift of $\mathcal{O}_{Y_0}$. Here we use the analytic nilpotent-extension criterion: a nilpotent ringed-space extension of a complex space whose successive ideal quotients are coherent analytic modules, with the induced local analytic algebra structures, is a complex-space thickening. In this construction those structures are obtained by lifting local coordinate functions to $(f_0)_*\mathcal{O}_{X_j}$; finite Taylor expansion in the nilpotent ideal defines the holomorphic functional calculus inherited from $\mathcal{O}_{X_j}$. Different coordinate lifts give compatible local analytic presentations of the same canonical sheaf of rings $\mathcal{B}_j$. The coherent nilpotent extensions therefore define complex-space thickenings Y_j and compatible morphisms $f_j : X_j \to Y_j$.

To check flatness correctly, one must use more than the displayed filtration. For any finite A_j-module M set $\mathcal{E}_M = \mathcal{O}_{X_j} \otimes_{A_j} M$. A composition series of M has factors $\mathbb{C}$; flatness of X_j/A_j and the long exact sequences show inductively that $R^1(f_0)_*\mathcal{E}_M = 0$. Consequently

$M \mapsto (f_0)_*\mathcal{E}_M$ is exact on finite A_j-modules. A finite presentation of M, together with right exactness of tensor product, identifies this functor with $M \mapsto \mathcal{B}_j \otimes_{A_j} M$. Thus $\mathcal{B}_j$ is A_j-flat. The same argument gives compatibility with reduction $A_j \to A_i$. We have obtained a flat formal target and a compatible formal morphism.

Analytic realization. Let $\mathcal{V} \to B$ be a semi-universal deformation of Y_0. Formal versality represents the target just constructed by a formal map $\widehat{\Delta} \to B$. The graphs of the f_j then define a formal map to the relative Douady space of $\mathcal{X} \times \mathcal{V}$ over $\Delta \times B$. Its closed point is $[\Gamma_{f_0}]$. Restrict to the open graph locus D^{gr} around this point, and retain the projection $D^{\mathrm{gr}} \to \Delta$. Analytic approximation, applied to the equations for a section of this projection, gives a holomorphic section σ through $[\Gamma_{f_0}]$ after shrinking Δ. Only approximation to order zero is needed to preserve this closed point; we do not assert that an arbitrary prescribed formal series converges. Nor is a section over the whole product $\Delta \times B$ required.

The map σ determines a holomorphic map $h : \Delta \to B$ and hence a proper flat family $\mathcal{Y} = h^*\mathcal{V}$. Pulling back the universal Douady family gives a family of graphs $\Gamma \subset \mathcal{X} \times_\Delta \mathcal{Y}$. On the graph locus the projection $\Gamma \to \mathcal{X}$ is an isomorphism, including its complex-space structure. Its inverse followed by the second projection defines F. Properness follows because $\mathcal{X} \to \Delta$ is proper and $\mathcal{Y} \to \Delta$ is separated.

Why the analytic extension is a contraction. Write $\mathcal{M} = F_*\mathcal{O}_\mathcal{X}$ and $\mathcal{N} = R^1F_*\mathcal{O}_\mathcal{X}$; both are coherent. Push forward $0 \to \mathcal{O}_\mathcal{X} \xrightarrow{t} \mathcal{O}_\mathcal{X} \to \mathcal{O}_{X_0} \to 0$. Since $R^1(f_0)_*\mathcal{O}_{X_0} = 0$, multiplication by t on $\mathcal{N}$ is surjective. Nakayama's lemma gives $\mathcal{N} = 0$ near Y_0. Properness of q allows us to shrink the disc so that this holds on all of $\mathcal{Y}$. The same long exact sequence now gives

$$0 \longrightarrow \mathcal{M} \xrightarrow{t} \mathcal{M} \longrightarrow \mathcal{O}_{Y_0} \longrightarrow 0.$$

In particular, the canonical map $u : \mathcal{O}_\mathcal{Y} \to \mathcal{M}$ is an isomorphism modulo t. Its coherent cokernel vanishes by Nakayama, so u is surjective near Y_0. Let $\mathcal{K} = \ker u$. Multiplication by t is injective on $\mathcal{M}$, so reduction modulo t preserves injectivity on the left in $0 \to \mathcal{K} \to \mathcal{O}_\mathcal{Y} \to \mathcal{M} \to 0$. As u modulo t is an isomorphism, $\mathcal{K}/t\mathcal{K} = 0$. Nakayama again gives $\mathcal{K} = 0$. Shrinking the disc once more proves $F_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$ everywhere. The image of the proper map F is closed; this direct-image equality forces it to be all of $\mathcal{Y}$. Thus F is the required contraction. ∎

Remark 2.2. The last paragraph applies to any proper analytic extension between the flat families in the theorem. It uses the vanishing on the central contraction, rather than an unproved assertion that connected fibres automatically persist in every deformation.

2.1 Directions suggested by the proof

The first question is whether the hypothesis $R^1(f_0)_*\mathcal{O}_{X_0} = 0$ can be weakened to the vanishing of the actual obstruction to deforming the map. This is a deformation-theoretic

question, not a consequence of the argument above. A useful test is to compare contractions with the same target but different $R^1(f_0)_*\mathcal{O}_{X_0}$. For higher-dimensional bases the elementary division by a single parameter must be replaced by a base-change and flatness argument. Finally, extension of a morphism should be distinguished from preservation of its MMP type: divisoriality, smallness and negativity impose additional conditions which this theorem does not assert.

3 Use in the minimal model program

The extension theorem constructs a map near a central fibre once the cohomological vanishing is known. Relative vanishing for a projective adjoint-negative contraction is a common way to supply it. After extension one must still check which curves are contracted, whether the map is small or divisorial, and whether the target is Kähler. Connectedness of fibres alone proves none of these assertions.

The threefold/fourfold MMP chapter combines analytic contraction with projective relative programs on auxiliary models. The transcendental base-point-free chapter addresses the further problem of descending a nef supporting class as a Kähler class on the contracted target. Thus the sequence of tasks is geometric: construct the target, extend the map when necessary, and then establish the positivity of the descended class.

References

[Art68] M. Artin, *On the solutions of analytic equations*, Invent. Math. **5** (1968), 277–291.

[Gra72] H. Grauert, *über die Deformation isolierter Singularitäten analytischer Mengen*, Invent. Math. **15** (1972), 171–198.

[KM92] J. Kollár and S. Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992), no. 3, 533–703.

Projectivity and Kählerness Criteria

Kähler MMP Yi Li

1 Introduction

The purpose of these notes is to prove that the Kähler minimal model program preserves the Kähler condition, in arbitrary dimension. We also prove a Kählerness criterion for Fujiki varieties in terms of rational curves, and collect the projectivity criteria that both arguments use.

A bimeromorphic contraction of a compact complex variety need not preserve projectivity, and need not preserve Kählerness. For the contractions arising in the minimal model program both are preserved, and without this the program could not be iterated. In dimension three the statement was known, by Höring–Peternell and by Das–Hacon–Yáñez, through Kleiman's Kählerness criterion, which fails on a general Fujiki variety. The argument given in Section 3 is dimension-free and does not use that criterion.

The setting is the following. For projective pairs the minimal model program is largely settled by the work of Birkar, Cascini, Hacon and McKernan.

Theorem 1.1 (Birkar–Cascini–Hacon–McKernan, [BCHM10]). *Let (X, B) be a klt pair with X projective and B big. If $K_X + B$ is pseudo-effective then (X, B) has a good minimal model; otherwise (X, B) has a Mori fibre space.*

Conjecture 1.2 (Kähler analogue). *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt pair with X compact Kähler and $B + \boldsymbol{\beta}_X$ modified big. If $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective then $(X, B + \boldsymbol{\beta})$ has a good minimal model; otherwise it has a Mori fibre space.*

Three things must be settled before Conjecture 1.2 can be approached: the cone and contraction theorems, which produce the object to be contracted; existence of flips; and the persistence of the hypotheses, that is, whether the conditions imposed on one step of the program still hold for its output. These notes treat the third, for the singularity, $\mathbb{Q}$-factoriality and Kählerness conditions.

Example 1.3. The persistence of Kählerness is not automatic for an arbitrary bimeromorphic contraction. On a generic quintic threefold, blow up a rational curve C_d of degree d whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, so that the exceptional divisor is a product of two rulings, and then contract the second ruling. The resulting variety need be neither Kähler nor projective; it is only Moishezon. What rules this out inside

the minimal model program is the negativity of the adjoint class on the contracted ray, and that is what the proofs below use.

Big picture

Each of the criteria below has the same form: a positivity property can fail only in the presence of a curve obstructing it, so ruling out that curve certifies the property. Section 3 proves that a flipping or divisorial contraction of a Kähler generalized pair has Kähler target; the difficulty is that the candidate class on the base exists as a Bott–Chern class only after the base is known to have rational singularities, which is precisely what the proof must produce. Section 4 establishes a dichotomy for compact Fujiki varieties that are not Kähler, and Section 5 collects the projectivity criteria of Kodaira, Claudon–Höring and Hacon–Xie.

Contents

1 Introduction **1**

2 Two criteria recalled **2**

3 The Kähler MMP preserves the Kähler condition **3**

4 A Kählerness criterion via rational curves **6**

5 Projectivity criteria **10**

5.1 Kodaira's criterion 10

5.2 The relative criterion of Claudon and Höring 10

5.3 The variant of Hacon and Xie 12

2 Two criteria recalled

We shall use the following two results, proved in the companion notes on rational curves. Both are stated for compact varieties in Fujiki's class $\mathcal{C}$, with no Kählerness and no $\mathbb{Q}$-factoriality assumed; this is essential below, where they are applied to a variety not yet known to be Kähler.

Theorem 2.1 (Hacon–Li–Xie, [HLX26, Theorem 3.6]). *Let X be a compact variety in Fujiki's class $\mathcal{C}$ such that $(X, B+\boldsymbol{\beta})$ is generalized klt and $B+\boldsymbol{\beta}_X$ is big. If $\alpha := [K_X + B + \boldsymbol{\beta}_X]$ is nef and NQC but not big, then X is covered by α-trivial rational curves.*

Theorem 2.2 (Hacon–Li–Xie, [HLX26, Theorem 3.7]). *Let X be a compact variety in Fujiki's class $\mathcal{C}$ such that $(X, B+\boldsymbol{\beta})$ is generalized klt. If $\alpha := [K_X + B + \boldsymbol{\beta}_X]$ is nef and big but not Kähler, then there exists an α-trivial rational curve $C \subset X$.*

In contrapositive form these are criteria: the absence of an α-trivial curve forces first bigness and then Kählerness of α.

3 The Kähler MMP preserves the Kähler condition

Theorem 3.1 (Hacon–Li–Xie, [HLX26]). *Let $(X, B+\boldsymbol{\beta})$ be a compact Kähler, generalized klt, strongly $\mathbb{Q}$-factorial generalized pair with $B+\boldsymbol{\beta}_X$ big, and let $f : X \to Z$ be a flipping or divisorial contraction. Then Z is Kähler.*

Idea of proof. Let $\alpha = K_X + B + \boldsymbol{\beta}_X + \omega$ be the nef class supporting the contracted extremal ray R, with ω Kähler. The proof falls into two halves. First one descends both the pair and the class: Stein-locally over Z the transcendental summand $\boldsymbol{\beta}_X + \omega$ is replaced by an $\mathbb{R}$-divisor Δ with $(X, B+\Delta)$ klt and $K_X+B+\Delta \equiv \alpha$, so that $K_X+B+\Delta \equiv_Z 0$; relative base point freeness, now applicable because Δ is a divisor, upgrades this to an $\mathbb{R}$-linear pullback formula, whence f is crepant, Z is klt and has rational singularities, and the Bott–Chern pullback lemma gives $\alpha = f^*\alpha_Z$. Second, one upgrades α_Z from nef to Kähler: were it not Kähler, Theorem 2.2 would produce an α_Z-trivial rational curve on Z, which lifts to an α-trivial curve not contracted by f.

Proof. By hypothesis there is a $(K_X + B + \boldsymbol{\beta}_X)$-negative extremal ray R, supported by a nef class

$$\alpha \;=\; K_X + B + \boldsymbol{\beta}_X + \omega, \qquad \omega \text{ Kähler},$$

with $\alpha^{\perp} \cap \overline{\mathrm{NA}}(X) = R$ and $f = \mathrm{cont}_R$. Here nefness of α is part of the input: the cone and contraction theorems produce the contraction together with its nef supporting class. It would be circular to deduce it from $\alpha = f^*\alpha_Z$, since nefness of α_Z will be derived from that of α.

Step 1: α is big. Put $\boldsymbol{\gamma} := \boldsymbol{\beta} + \bar{\omega}$. Then $(X, B+\boldsymbol{\gamma})$ is generalized klt, $B+\boldsymbol{\gamma}_X$ is big, and $\alpha = K_X + B + \boldsymbol{\gamma}_X$, so Theorem 2.1 applies. If α were not big, X would be covered by a family $\{C_t\}$ of α-trivial rational curves. Each C_t satisfies $[C_t] \in \alpha^{\perp} \cap \overline{\mathrm{NA}}(X) = R$, hence is contracted by f and contained in $\mathrm{Ex}(f)$, a proper closed subset of X. A covering family cannot lie in a proper closed subset, so α is big. Note that the covering strength

of Theorem 2.1 is what is used here: the existence of a single α-trivial curve is no contradiction, since $\mathrm{Ex}(f)$ carries such curves by construction.

Step 2: reduction to a log pair. The assertion is local over Z, so we may assume Z relatively compact Stein. Since f is bimeromorphic it is Moishezon, so there is a log resolution $\nu : X' \to X$ over Z with $X' \to Z$ projective. As $X' \to Z$ is bimeromorphic, $\boldsymbol{\beta}_{X'} + \nu^*\omega$ is nef and big over Z, and hence

$$\boldsymbol{\beta}_{X'} + \nu^*\omega \;\equiv\; \Delta', \qquad (X', B' + \Delta') \text{ sub-klt};$$

the pair upstairs is only sub-klt because the crepant pullback B' may have negative coefficients. Pushing forward, $\boldsymbol{\beta}_X + \omega \equiv \Delta$ with $(X, B + \Delta)$ klt. In particular X has rational singularities.

Step 3: $K_X + B + \Delta \equiv_Z 0$. Since $\Delta \equiv \boldsymbol{\beta}_X + \omega$ we have

$$K_X + B + \Delta \;\equiv\; K_X + B + \boldsymbol{\beta}_X + \omega \;=\; \alpha.$$

The curves contracted by $f = \mathrm{cont}_R$ are exactly those with class in R, and α supports R, so $\alpha \cdot C = 0$ for every such curve; that is, $K_X + B + \Delta \equiv_Z 0$.

> **Remark 3.2.** The contraction f is $(K_X + B + \boldsymbol{\beta}_X)$-negative, so $(K_X + B + \boldsymbol{\beta}_X) \cdot C < 0$ for contracted curves, and it is the identity
>
> $$(K_X + B + \Delta) \cdot C \;=\; (K_X + B + \boldsymbol{\beta}_X) \cdot C + \omega \cdot C$$
>
> together with the fact that the two terms cancel on R that gives Step 3. It is therefore essential that Δ absorb the Kähler class ω and not merely the moduli part $\boldsymbol{\beta}_X$: otherwise $K_X + B + \Delta$ would be f-negative, and the log canonical model of Step 4 would not be Z.

Step 4: the log canonical model of $(X, B + \Delta)$ over Z is Z. Apply the relative base point free theorem to $f : X \to Z$ and $H := K_X + B + \Delta$. Its hypotheses hold: $(X, B + \Delta)$ is klt by Step 2; H is f-nef, indeed $H \equiv_Z 0$ by Step 3; and $H - (K_X + B + \Delta) = 0$ is f-nef and f-big, the latter since f is bimeromorphic. We obtain a factorisation

$$f : X \xrightarrow{\varphi} W \xrightarrow{g} Z, \qquad g \text{ projective}, \quad \varphi^*A = K_X + B + \Delta,$$

with A a g-ample $\mathbb{R}$-Cartier class on W. Now φ contracts precisely those f-vertical curves C with $(K_X + B + \Delta) \cdot C = 0$, which by Step 3 is every f-contracted curve. Hence g is a bimeromorphic morphism contracting no curve, so it is finite; as Z is normal, $W = Z$ and $\varphi = f$.

Step 5: f is crepant and Z has rational singularities. By Step 4, $K_X + B + \Delta = f^*A$ with A an $\mathbb{R}$-Cartier class on Z. Since f is bimeromorphic, the projection formula gives

$f_*f^*A = A$, while $f_*(K_X + B + \Delta) = K_Z + B_Z + \Delta_Z$ with $B_Z := f_*B$ and $\Delta_Z := f_*\Delta$. Therefore $A = K_Z + B_Z + \Delta_Z$ is $\mathbb{R}$-Cartier and

$$K_X + B + \Delta \ \sim_{\mathbb{R}} \ f^*(K_Z + B_Z + \Delta_Z). \tag{1}$$

Thus f is crepant, $(Z, B_Z + \Delta_Z)$ is klt, and Z has rational singularities.

Remark 3.3. Numerical triviality over Z alone would not suffice. The relation $\equiv_Z 0$ carries no information about $K_Z + B_Z + \Delta_Z$ being $\mathbb{R}$-Cartier, and without $\mathbb{R}$-Cartierness there is no pair on Z, hence no discrepancies, no kltness and no rational singularities. The passage from $\equiv_Z 0$ to (1) is exactly what base point freeness supplies, and it is available only because Δ is a divisor. In the divisorial case one might expect a discrepancy term $+aE$ in (1); its absence is the content of crepancy, and combined with kltness of $(X, B + \Delta)$ it gives $a(E; Z, B_Z + \Delta_Z) = -\operatorname{coeff}_E(B + \Delta) > -1$.

Step 6: α descends. Recall the pullback lemma for Bott–Chern classes: if $f : X \to Z$ is a proper bimeromorphic morphism with connected fibres between normal compact complex spaces in Fujiki's class $\mathcal{C}$ with rational singularities, then $f^* : H^{1,1}_{\mathrm{BC}}(Z) \to H^{1,1}_{\mathrm{BC}}(X)$ is injective with image

$$\{\eta \ : \ \eta \cdot C = 0 \ \text{ for every curve } C \text{ with } f_*C = 0\}.$$

All hypotheses are available: X is Kähler and Z is its bimeromorphic image, so both lie in class $\mathcal{C}$, while Z is not yet known to be Kähler; X has rational singularities by Step 2 and Z by Step 5; and α is trivial on every f-contracted curve by Step 3. Hence there is a unique $\alpha_Z \in H^{1,1}_{\mathrm{BC}}(Z)$ with $\alpha = f^*\alpha_Z$, and pushing forward, $\alpha_Z = K_Z + B_Z + \boldsymbol{\beta}_Z + f_*\omega$. Applying crepancy to the generalized pair, $(Z, B_Z + \boldsymbol{\beta}_Z + f_*\omega)$ is generalized klt.

Remark 3.4. Rational singularities of Z is not a technical convenience here. The proof of the pullback lemma runs the exponential-type sequence $0 \to \mathbb{R} \to \mathcal{O} \to \mathcal{H} \to 0$ together with a Leray argument, and injectivity uses $H^1(Z, \mathcal{O}_Z) \cong H^1(\hat{Z}, \mathcal{O}_{\hat{Z}})$ for a resolution $\hat{Z} \to Z$, which is the definition of rational singularities. This is why Step 5 must precede Step 6. Note also that the correct b-class for the generalized pair is $\boldsymbol{\beta} + \bar{\omega}$, taken from X, rather than $\overline{f_*\omega}$ taken from Z: the latter would be b-nef only if $f_*\omega$ were nef on Z, whereas $f_*\omega$ is big and not nef.

Step 7: α_Z is nef and big. Let $C \subset Z$ be an irreducible curve and choose an irreducible $C' \subset X$ with $f(C') = C$, say of degree $m > 0$ over C. By the projection formula,

$$\alpha_Z \cdot C \ = \ \tfrac{1}{m}\, \alpha_Z \cdot f_*C' \ = \ \tfrac{1}{m}\, f^*\alpha_Z \cdot C' \ = \ \tfrac{1}{m}\, \alpha \cdot C' \ \geq \ 0,$$

so α_Z is nef; surjectivity of f is what makes this a descent, rather than the trivial statement that a pullback of a nef class is nef. Bigness is preserved by pushforward under a proper bimeromorphic morphism, so $\alpha_Z = f_*\alpha$ is big by Step 1.

Step 8: no curve on Z is α_Z-trivial. Suppose $\alpha_Z \cdot C = 0$ for some irreducible curve $C \subset Z$, and choose C' as in Step 7. Then

$$0 \;=\; m\,(\alpha_Z \cdot C) \;=\; \alpha_Z \cdot f_* C' \;=\; f^*\alpha_Z \cdot C' \;=\; \alpha \cdot C',$$

so C' is α-trivial, hence $[C'] \in R$ and f contracts C', contradicting $f(C') = C$.

Step 9: conclusion. By Steps 6 and 7 the class α_Z is the adjoint class of the generalized klt pair $(Z, B_Z + \boldsymbol{\beta}_Z + f_*\omega)$ on the compact Fujiki variety Z, and it is nef and big. If it were not Kähler, Theorem 2.2 would give an α_Z-trivial rational curve on Z, contradicting Step 8. Hence α_Z is Kähler and Z is Kähler. ∎

Remark 3.5. Both dichotomies invoked above, in Step 1 and in Step 9, are closed by the same observation: f has already contracted every α-trivial curve. The substance of the argument therefore lies in Steps 2–6, which manufacture on Z the hypotheses of Theorem 2.1 and Theorem 2.2; the two theorems themselves are full minimal model program arguments carried out on a Fujiki, rather than Kähler, variety.

4 A Kählerness criterion via rational curves

We turn to the following dichotomy for compact Fujiki varieties, an analogue of which for Moishezon spaces is due to Villalobos-Paz.

Conjecture 4.1 (Hacon–Li–Xie, [HLX26]). *Let $(X, B + \boldsymbol{\beta})$ be a compact generalized klt pair in Fujiki's class $\mathcal{C}$. If X is not Kähler, then either*

1. *X contains a rational curve C with $-[C] \in \overline{\mathrm{NA}}(X)$, or*
2. *there exists a small $\mathbb{Q}$-factorial Kähler modification $\mu : X^{\mathrm{qf}} \to X$.*

Notation 4.2. Throughout this section $\nu : X' \to X$ denotes a Kähler resolution with Kähler class ω', and $D := K_{X'} + B' + \boldsymbol{\beta}_{X'}$. We write

$$t \;:=\; \inf\{\tau \geq 0 \;\mid\; D + \tau\,\omega' \text{ is nef over } X\}$$

for the **relative nef threshold**, $\alpha := D + t\,\omega'$, and $\psi : X' \to \bar{X}$ for the log canonical model of α over X. Let $F \subset \overline{\mathrm{NA}}(X'/X)$ denote the face spanned by the ψ-contracted curves; for very general ω' it is a single ray $\mathbb{R}^+[C]$.

Idea of proof. If $t = 0$ then the relative negative part of D vanishes, the resolution is small, and (2) holds. If $t > 0$ one forms the log canonical model at the threshold, for which no minimal model program is needed since α is nef and big over X, hence relatively semiample. The heart of the argument is then Lemma 4.3: the contracted ray

either releases a rational curve realising (1), or is extremal in the *absolute* Mori cone. In the latter case Theorem 3.1 makes $\bar{X}$ Kähler and the threshold drops strictly; finiteness of models forces $t = 0$ after finitely many steps.

Lemma 4.3. *In the situation of Notation 4.2, assume $t > 0$ and let $\mathbb{R}^+[C] = F$. Then either X contains a rational curve R with $-[R] \in \overline{\mathrm{NA}}(X)$, or $\mathbb{R}^+[C]$ is a D-negative extremal ray of the absolute cone $\overline{\mathrm{NA}}(X')$.*

Proof. Assume $\mathbb{R}^+[C]$ is not extremal in $\overline{\mathrm{NA}}(X')$. For $0 < \delta < t$ set

$$\alpha_\delta \;:=\; D + (t-\delta)\,\omega' \;=\; \alpha - \delta\,\omega'.$$

This is again the adjoint class of a generalized klt Kähler pair on X', with $(t-\delta)\omega'$ playing the role of the Kähler summand, so the cone theorem gives

$$\overline{\mathrm{NA}}(X') \;=\; \overline{\mathrm{NA}}(X')_{\alpha_\delta \geq 0} \;+\; \sum_{R \in S_\delta} R, \qquad S_\delta := \{\text{extremal rays } R \;:\; \alpha_\delta \cdot R < 0\},$$

with S_δ finite and each of its members spanned by a rational curve.
If $z \in \overline{\mathrm{NA}}(X') \setminus \{0\}$ satisfies $\alpha_\delta \cdot z \geq 0$, then

$$\alpha \cdot z \;=\; \alpha_\delta \cdot z + \delta\,\omega' \cdot z \;\geq\; \delta\,\omega' \cdot z \;>\; 0,$$

since a Kähler class is strictly positive on $\overline{\mathrm{NA}}(X') \setminus \{0\}$. Moreover for $\delta' < \delta$ we have $\alpha_\delta = \alpha_{\delta'} - (\delta - \delta')\omega'$, so $\alpha_{\delta'} \cdot R < 0$ implies $\alpha_\delta \cdot R < 0$: the finite sets S_δ shrink as δ decreases, hence stabilise, with

$$\bigcap_{\delta > 0} S_\delta \;=\; \{R \;:\; \alpha \cdot R \leq 0\}.$$

Combining, for all sufficiently small δ we may write

$$[C] \;=\; \sum_{i=1}^{k} \sigma_i [\Sigma_i] + \gamma, \qquad \sigma_i > 0, \tag{2}$$

where each Σ_i is a rational curve spanning an extremal ray of $\overline{\mathrm{NA}}(X')$ with $\alpha \cdot \Sigma_i \leq 0$, and γ is either 0 or satisfies $\alpha \cdot \gamma > 0$. Each such Σ_i is automatically D-negative, since $D \cdot \Sigma_i = \alpha \cdot \Sigma_i - t\,\omega' \cdot \Sigma_i < 0$; this is why the cone theorem detects it and why it may be taken rational.
We claim $k \geq 1$. Indeed $[C] \neq 0$, so if the sum in (2) were empty we would have $[C] = \gamma \neq 0$ and hence $\alpha \cdot C > 0$; but $\alpha \cdot C = \psi^* \bar{\alpha} \cdot C = \bar{\alpha} \cdot \psi_* C = 0$.
Suppose some Σ_i is not ν-vertical. Since C is ψ-contracted it is ν-vertical, so $\nu_*[C] = 0$, and pushing (2) forward gives $0 = \sum_j \sigma_j [\nu_* \Sigma_j] + \nu_* \gamma$. Solving for the i-th term,

$$-[\nu_* \Sigma_i] \;=\; \sum_{j \neq i} \frac{\sigma_j}{\sigma_i} [\nu_* \Sigma_j] + \frac{1}{\sigma_i}\,\nu_* \gamma \;\in\; \overline{\mathrm{NA}}(X),$$

because ν_* maps $\overline{\mathrm{NA}}(X')$ into $\overline{\mathrm{NA}}(X)$. As Σ_i is rational and not contracted, $R := \nu_*\Sigma_i$ is a rational curve on X with $-[R] \in \overline{\mathrm{NA}}(X)$, which is the first alternative.
Otherwise every Σ_i is ν-vertical, so $[\Sigma_i] \in \overline{\mathrm{NA}}(X'/X)$. Since α is nef over X we get $\alpha\cdot\Sigma_i \geq 0$, and with $\alpha\cdot\Sigma_i \leq 0$ from (2) this forces $\alpha\cdot\Sigma_i = 0$. Hence $[\Sigma_i] \in \alpha^\perp \cap \overline{\mathrm{NA}}(X'/X) = F = \mathbb{R}^+[C]$, so $\mathbb{R}^+[\Sigma_1] = \mathbb{R}^+[C]$; but $\mathbb{R}^+[\Sigma_1]$ is extremal in $\overline{\mathrm{NA}}(X')$, contrary to assumption. ∎

Remark 4.4. The perturbation $\alpha \rightsquigarrow \alpha_\delta$ serves one purpose. Applied to α itself, the cone theorem produces only rays on which α is strictly negative, so the sum would be empty precisely when α is nef, and the conclusion would be vacuous in the case of interest. Passing to α_δ converts the condition "α-trivial or α-negative" into the single condition "α_δ-negative", which the cone theorem detects, and finiteness of the S_δ makes the limit $\delta \to 0$ harmless. Nefness of α over X enters only once, in the last paragraph.

Proposition 4.5. *In the situation of Notation 4.2 with $t > 0$, suppose $\mathbb{R}^+[C]$ is extremal in $\overline{\mathrm{NA}}(X')$. Then either X contains a curve R with $-[R] \in \overline{\mathrm{NA}}(X)$, or $\bar{X}$ is Kähler.*

Proof. Write $\Gamma := \mathbb{R}^+[C]$. The supporting class to be used is not $\alpha = D + t\,\omega'$, which was never assumed absolutely nef, but the nef class α_Γ furnished by the existence theorem for nef supporting classes of a negative extremal ray: $\alpha_\Gamma - D$ is Kähler and $\alpha_\Gamma^\perp \cap \overline{\mathrm{NA}}(X') = \Gamma$. That theorem is proved by a compactness argument on the unit sphere of $N_1(X')$ which never refers to the target of the contraction, so it may be applied before $\bar{X}$ is known to be Kähler. A curve on X' is α_Γ-trivial exactly when its class lies in Γ, so cont_Γ contracts precisely the α_Γ-trivial curves, and Theorem 3.1 applies: the target of cont_Γ is Kähler.

It remains to identify cont_Γ with ψ. Both contract Γ, so cont_Γ factors as $X' \xrightarrow{\psi} \bar{X} \xrightarrow{h} Y$, and the two coincide if and only if every curve with class in Γ is ν-vertical. Let Σ be a curve with $[\Sigma] = \lambda[C]$, $\lambda > 0$. If Σ is ν-vertical then $[\Sigma] \in \alpha^\perp \cap \overline{\mathrm{NA}}(X'/X) = F$, so Σ is ψ-contracted. If not, then $\nu_*\Sigma$ is a curve on X with $[\nu_*\Sigma] = \lambda\,[\nu_*C] = 0$, so $-[\nu_*\Sigma] = 0 \in \overline{\mathrm{NA}}(X)$ and the first alternative holds in the strong form of a numerically trivial curve. In the remaining case h contracts no curve and is therefore an isomorphism, so $\bar{X}$ is Kähler. ∎

Remark 4.6. The coincidence of the two rays does not by itself identify the two contractions. That inference is valid whenever being contracted is detected by the numerical class, which is what an absolutely positive supporting class provides and is why the point is automatic in the projective setting. Here the supporting class of ψ is Kähler only over X, so on X' it detects only the ν-vertical curves, and it is the dichotomy above that breaks the circle.
One point remains open. Conclusion (1) of Conjecture 4.1 asks for a rational curve, whereas Proposition 4.5 produces a rational curve only when the escaping Σ may be taken rational. Closing this requires rational chain connectedness of the fibres of a

negative extremal contraction in the compact Kähler generalized klt setting; failing that, conclusion (1) must be stated with "curve" in place of "rational curve".

Proof of Conjecture 4.1. *Step 1: normalisation of the resolution.* As X is Fujiki, choose a Kähler resolution $\nu : X' \to X$ with Kähler class ω' as in Notation 4.2, arranged so that $\operatorname{Supp} N(D/X) = \operatorname{Ex}(\nu)$. Writing

$$K_{X'} + B' + \boldsymbol{\beta}_{X'} \;=\; \nu^*(K_X + B + \boldsymbol{\beta}_X) + G, \qquad B' \;=\; \nu_*^{-1}(B) + (1-\epsilon)\operatorname{Ex}(\nu)$$

for $0 < \epsilon \ll 1$, the pair upstairs is klt, as required by the minimal model program arguments below, and $G \geq 0$ has support exactly $\operatorname{Ex}(\nu)$.
Step 2: the case $t = 0$. Here D is nef over X, so $N(D/X) = 0$; by Step 1 the support of the relative negative part is $\operatorname{Ex}(\nu)$, so ν contracts no divisor and is small. Conclusion (2) then follows from Step 6 below.
Step 3: the case $t > 0$. The class $\alpha = D + t\,\omega'$ is nef over X by definition of t, and big over X since ν is bimeromorphic; hence it is semiample over X and the log canonical model $\psi : X' \to \bar{X}$ over X exists without recourse to a minimal model program, with $K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} + t\,\bar{\omega}$ Kähler over X. For very general ω' the face F is a single ray $\mathbb{R}^+[C]$ with $[C] \neq 0$. By Lemma 4.3 and Proposition 4.5 we may assume conclusion (1) fails, so that $\bar{X}$ is Kähler.
Step 4: the flip or the divisorial contraction. Let $X^+ \to \bar{X}$ be the log canonical model of D over $\bar{X}$, which exists because $-D$ is Kähler over $\bar{X}$. If ψ is divisorial then $X^+ = \bar{X}$; if ψ is small then $X^+ \to \bar{X}$ is the flip of ψ. Taking the model of D rather than of $D + (t-\epsilon)\omega'$ is harmless: over $\bar{X}$ one has $D \equiv_{\bar{X}} -t\,\omega'$ and $D + (t-\epsilon)\omega' \equiv_{\bar{X}} -\epsilon\,\omega'$, which are positive multiples of one another, and log canonical models depend on the relative class only up to positive scaling.
Step 5: the threshold drops. From $\psi^*(K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} + t\,\bar{\omega}) = D + t\,\omega'$ one obtains the corresponding identity on X^+, and hence, for $0 < t - s \ll 1$,

$$K_{X^+} + B^+ + \boldsymbol{\beta}_{X^+} + s\,\omega^+ \;=\; f^*\big(K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} + t\,\bar{\omega}\big) - (t-s)\,\omega^+,$$

the sum of the pullback of an absolutely Kähler class and an f-Kähler class, hence absolutely Kähler. Kählerness over X would yield only the inequality $t^+ \leq s < t$; absolute Kählerness yields this and in addition keeps X^+ Kähler, which is what permits the argument to be repeated. Replacing $(X', B' + \boldsymbol{\beta}_{X'} + \omega')$ by $(X^+, B^+ + \boldsymbol{\beta}_{X^+} + s\,\omega^+)$ and iterating produces a strictly decreasing sequence of thresholds $t_0 > t_1 > t_2 > \cdots$ with pairwise distinct log canonical models.
Step 6: termination and conclusion. Termination is checked locally over X: cover X by relatively compact Stein open sets, realise ω' there by an ample $\mathbb{R}$-divisor, trade the moduli part for a boundary, and apply finiteness of weak log canonical models. Hence $t_n = 0$ for some n, and $X_n \to X$ is small by Step 2. Let $\mu : X^{\mathrm{qf}} \to X_n$ be a small strongly $\mathbb{Q}$-factorial modification. Since μ is projective and X_n is Kähler, the class $\mu^*(\text{Kähler}) + (\mu\text{-ample})$ is Kähler on X^{qf}. The composite $X^{\mathrm{qf}} \to X_n \to X$ is therefore small, strongly $\mathbb{Q}$-factorial and Kähler, which is conclusion (2). ∎

5 Projectivity criteria

In this section X denotes a normal compact Kähler space, $f : X \to Y$ a fibration, $\Omega^{[2]}$ the sheaf of reflexive 2-forms, and $d := \dim X - \dim Y$.

5.1 Kodaira's criterion

Proposition 5.1 ([CH24, Proposition 2.1]). *Let X be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$. Then X is projective.*

Under Hodge symmetry the hypothesis reads $H^0(X, \Omega^2_X) = 0$.

Proof. Suppose first that X is a compact Kähler manifold. Since $H^2(X, \mathcal{O}_X) = 0$, the Hodge decomposition collapses to $H^2(X, \mathbb{R}) = H^{1,1}_{\mathrm{BC}}(X)$, so rational classes are dense among the real $(1,1)$-classes. The Kähler cone is open, hence contains a rational class, which by the exponential sequence

$$H^1(X, \mathcal{O}^*_X) \longrightarrow H^2(X, \mathbb{R}) \longrightarrow H^2(X, \mathcal{O}_X) = 0$$

lifts to a $\mathbb{Q}$-line bundle; this bundle is ample by the Kodaira embedding theorem. In the singular case, pass to a resolution: once the resolution is projective, X is a Kähler Moishezon variety with rational singularities, hence projective. ∎

Remark 5.2. The vanishing hypothesis cannot be dropped. In general $H^{1,1}_{\mathrm{BC}}(X)$ is a proper subspace of $H^2(X, \mathbb{R})$ and may contain no nonzero rational class at all, much as an irrational line in the plane meets the rational lattice only at the origin. The converse also fails: a projective K3 surface has $H^2(X, \mathcal{O}_X) \neq 0$.

5.2 The relative criterion of Claudon and Höring

Theorem 5.3 ([CH24, Theorem 1.1]). *Let $f : X \to Y$ be a fibration between compact Kähler manifolds. Assume that either*

(a.1) $f^* : H^0(Y, \Omega^2_Y) \to H^0(X, \Omega^2_X)$ is an isomorphism, or

(a.2) f is Moishezon, that is, there is a line bundle L on X which is f-big.

Then f is a projective morphism.

Theorem 5.4 ([CH24, Theorem 3.1]). *Let $f : X \to Y$ be a fibration between normal compact Kähler spaces, with X having strongly $\mathbb{Q}$-factorial klt singularities. Assume that either*

(a.1) Y has klt singularities and $f^* : H^0(Y, \Omega^{[2]}_Y) \to H^0(X, \Omega^{[2]}_X)$ is an isomorphism, or

(a.2) f is Moishezon.
Then f is a projective morphism.

Idea of proof. Case (a.1) for smooth X and Y is a direct construction: one takes a rational class near a Kähler one, uses (a.1) to see that its $(2,0)$-part is a pullback, subtracts a rational pullback obtained from a pushforward formula, and is left with a rational $(1,1)$-class that is still relatively Kähler. Case (a.2) resolves f projectively and descends a relative minimal model program, checking that projectivity over Y survives each step. The singular statement is reduced to these two.

Proof of Theorem 5.3, case (a.1). Fix a Kähler class ω on X. Since Y is compact and relative Kählerness is an open condition, we may choose a rational class $\eta \in H^2(X,\mathbb{Q})$ close enough to ω that its $(1,1)$-part remains relatively Kähler. Hypothesis (a.1) gives the Hodge decomposition

$$\eta \;=\; f^*(\gamma) + \beta + f^*(\bar\gamma), \qquad \gamma \in H^{2,0}(Y), \tag{3}$$

with β of type $(1,1)$; restricting to a fibre kills the pullback terms, so $\eta|_{X_y} = \beta|_{X_y}$.
The pushforward f_* is a morphism of Hodge structures of bidegree $(-d,-d)$, so it annihilates classes of degree less than $2d$. Expanding η^d by (3) and applying the projection formula, every term with at least one pullback factor contributes $f_*(\beta^{d-j})$ in negative degree, whence $f_*(\eta^d) = f_*(\beta^d)$. For η^{d+1} only the terms with $j = 0, 1$ survive:

$$f_*(\eta^{d+1}) \;=\; f_*(\beta^{d+1}) + (d+1)\, f_*(\eta^d)\,(\gamma + \bar\gamma). \tag{4}$$

This is the Hodge decomposition of $f_*(\eta^{d+1})$ in $H^2(Y)$: the class β^{d+1} has type $(d{+}1, d{+}1)$, so its pushforward is of type $(1,1)$, while $\beta^d \wedge \gamma$ has type $(d{+}2, d)$, so its pushforward is of type $(2,0)$, and symmetrically for $\bar\gamma$.
Since f_* is defined over $\mathbb{Q}$ and η is rational, $f_*(\eta^d) = \int_F \beta^d$ is a positive rational number, positive because β restricts to a Kähler class on the general fibre F. Set

$$\eta_Y \;:=\; \frac{1}{(d+1)\, f_*(\eta^d)}\, f_*\Big(\eta^{d+1}\Big) \in H^2(Y,\mathbb{Q}), \qquad \tilde\eta \;:=\; \eta - f^*(\eta_Y).$$

Substituting (4), the term in $\gamma + \bar\gamma$ cancels the corresponding term of (3), leaving

$$\tilde\eta \;=\; \beta - \frac{1}{(d+1) f_*(\eta^d)}\, f^*\big(f_*(\beta^{d+1})\big),$$

a rational class of type $(1,1)$ with $\tilde\eta - \eta \in f^*H^2(Y,\mathbb{Q})$.
By the Lefschetz theorem on $(1,1)$-classes, $m\tilde\eta = c_1(L)$ for some $L \in \mathrm{Pic}(X)$ and m sufficiently divisible. As $\tilde\eta$ differs from η by a pullback and β is relatively Kähler, $c_1(L)$ is relatively Kähler; by Fujiki's lemma there is a Kähler class $[\omega_Y]$ on Y with $c_1(L) + f^*[\omega_Y]$ Kähler, and the relative ampleness criterion then shows that L is f-ample. Hence f is projective. ∎

Proof of Theorem 5.3, case (a.2). By Chow's lemma there is a log resolution $\mu_0 : X_0 \to X$ with both μ_0 and $f \circ \mu_0$ projective. Decompose the bimeromorphic map μ_0 into a sequence of divisorial contractions and flips over X, and write $\mu_i : X_i \to X$ and $f_i := f \circ \mu_i$. Then f_0 is projective and $f_m \simeq f$, so it suffices to show that projectivity over Y is preserved by a single step.
Let $\varphi : X_0 \to Z$ be the elementary Mori contraction of an extremal ray $\mathbb{R}^+[C] \subset \overline{\mathrm{NE}}(X_0/X)$. We claim this ray remains extremal in the larger cone $\overline{\mathrm{NE}}(X_0/Y)$. Let $l_1, l_2 \in \overline{\mathrm{NE}}(X_0/Y)$ with $l_1 + l_2 \in \mathbb{R}^+[C]$. Applying φ_*, which annihilates $[C]$, gives $\varphi_* l_1 + \varphi_* l_2 = 0$ in $\overline{\mathrm{NE}}(Z)$. Since Z is Kähler, a Kähler class is strictly positive on $\overline{\mathrm{NE}}(Z) \setminus \{0\}$, so this cone is pointed and $\varphi_* l_1 = \varphi_* l_2 = 0$; as $\ker \varphi_* = \mathbb{R}[C]$ we conclude $l_j \in \mathbb{R}^+[C]$, proving the claim.
The fundamental theorems now provide a projective contraction $\eta : X_0 \to \tilde{X}_1$ with $\tilde{X}_1 \to Y$ projective. Since φ and η contract the same curves, the rigidity lemma gives an isomorphism $\tilde{X}_1 \xrightarrow{\sim} Z$ identifying them. If φ is divisorial then $Z \simeq X_1$ and $X_1 \to Y$ is projective; if φ is small then the flip $\varphi^+ : X_1 \to Z$ is projective, being polarised by K_{X_1}, which is relatively ample over Z, so again $X_1 \to Y$ is projective. The result follows by induction on the number of steps. ∎

Proof of Theorem 5.4. Let $\mu_Y : Y' \to Y$ and $\mu_X : X' \to X$ be projective modifications by compact Kähler manifolds inducing a fibration $f' : X' \to Y'$. Reflexive forms are invariant under resolution, so

$$H^0(Y', \Omega^2_{Y'}) \simeq H^0(Y, \Omega^{[2]}_Y), \qquad H^0(X', \Omega^2_{X'}) \simeq H^0(X, \Omega^{[2]}_X),$$

and the always-injective map $(f')^*$ inherits surjectivity from the hypothesis, hence is an isomorphism. By case (a.1), f' is projective. Consequently f is Moishezon: pushing a relatively ample line bundle on X' down to X yields a relatively big line bundle, the pushforward being a line bundle because X is strongly $\mathbb{Q}$-factorial. Case (a.2) now applies. ∎

Remark 5.5. Taking Y to be a point in (a.1) recovers Proposition 5.1: the source of the map is zero, so the hypothesis reads $H^0(X, \Omega^2_X) = 0$. Injectivity of f^* is automatic for a fibration, so the content of the hypothesis is surjectivity, that is, that every holomorphic 2-form on X descends. The two alternatives are relative forms of two different classical statements: (a.1) of Kodaira's criterion, (a.2) of the fact that a Moishezon Kähler variety is projective. This also explains why the singular version requires Y klt in (a.1) but not in (a.2): reflexive 2-forms behave like forms on a resolution only under klt hypotheses, and (a.1) alone is Hodge-theoretic.

5.3 The variant of Hacon and Xie

The following criterion concerns a Moishezon morphism from a Kähler variety with strongly $\mathbb{Q}$-factorial klt singularities, and imposes no Kähler hypothesis on the target.

Lemma 5.6 (Hacon–Xie, [HX26, Lemma 2.40]). *Let $f : X \to Z$ be a proper morphism between compact complex spaces, with X a strongly $\mathbb{Q}$-factorial klt Kähler variety. Suppose there is a projective bimeromorphic morphism $h : W \to X$ with W a Kähler manifold such that $W \to X \to Z$ is projective. Then f is projective. In particular, a strongly Moishezon f is projective in this setting.*

Proof. Put $X_0 := W$ and $\mu_0 := h$, so that μ_0 and $f \circ \mu_0$ are projective. For $0 < \epsilon \ll 1$ write

$$K_{X_0} + (1-\epsilon)\,\mathrm{Ex}(\mu_0) \;=\; \mu_0^* K_X + F,$$

with $F \geq 0$ exceptional and $\mathrm{Supp}(F) = \mathrm{Ex}(\mu_0)$, so that a minimal model program for this boundary over X contracts only μ_0-exceptional curves and terminates at X. Running it gives

$$W = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X,$$

with each X_i projective over the Kähler variety X, hence Kähler. Let $\varphi : X_i \to Y_i$ be the extremal contraction of the i-th step, so that Y_i is Kähler as well.
We first check that the contracted ray $\mathbb{R}^+[C_i]$, extremal in $\overline{\mathrm{NE}}(X_i/X)$, remains extremal in $\overline{\mathrm{NE}}(X_i/Z)$. If not, write $[C_i] = l_1 + l_2$ with l_1, l_2 linearly independent in $\overline{\mathrm{NE}}(X_i/Z)$. Pushing forward gives $\varphi_* l_1 + \varphi_* l_2 = 0$ in $\overline{\mathrm{NE}}(Y_i)$, a pointed cone since Y_i is Kähler, so $\varphi_* l_1 = \varphi_* l_2 = 0$; but then l_1, l_2 lie in $\overline{\mathrm{NE}}(X_i/X)$, contradicting extremality there.
Now suppose $X_i \to Z$ is projective. By the previous paragraph φ contracts an extremal ray of $\overline{\mathrm{NE}}(X_i/Z)$ on the Kähler variety X_i, so the relative cone theorem over Z yields a contraction $c_i : X_i \to Y_i'$ with $Y_i' \to Z$ projective, characterised by contracting exactly the curves with class in $\mathbb{R}^+[C_i]$. Both φ and c_i are proper with connected fibres onto normal spaces and contract the same curves, so by rigidity each factors through the other; the induced map $Y_i \to Y_i'$ contracts no curve and has connected fibres, hence is finite and bimeromorphic onto a normal space, hence an isomorphism. Thus $c_i = \varphi$ and $Y_i \to Z$ is projective, whence $X_{i+1} \to Z$ is projective: in the divisorial case $X_{i+1} = Y_i$, and in the flipping case $X_{i+1} \to Y_i$ is the flip, projective over Y_i. Since $X_0 \to Z$ is projective by hypothesis, induction gives that $X_n = X \to Z$ is projective. ∎

Remark 5.7. This is the form in which the Kähler minimal model program applies the criterion, a log resolution being what is available in practice. The hypothesis that W be Kähler, and not merely Moishezon, is used exactly once, to force every Y_i to be Kähler; the argument on extremality rests on the Mori cone of Y_i being pointed, which can fail for a Moishezon variety.

References

[BCHM10] C. Birkar, P. Cascini, C.D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society **23** (2010), no. 2, 405–468.

[CH24] Benoît Claudon and Andreas Höring, *Projectivity criteria for kähler morphisms*, 2024.

[HLX26] C. D. Hacon, Y. Li, and L. Xie, *Fujiki class $\mathcal{C}$ varieties and a kähler criterion*, 2026.

[HX26] C. D. Hacon and L. Xie, *On the kähler mmp and the transcendental base-point-free theorem*, 2026.

Preservation of the Kähler Condition

Kähler MMP

Yi Li

Contents

1 The local model: repairing a pullback 1

2 The contraction theorem needed in dimension three 2

3 Why curve positivity alone is not a proof 2

4 Research directions 2

Big picture

The Kähler condition is part of the output of every MMP step. Being bimeromorphic to a Kähler space only places a space in Fujiki's class; it does not furnish a Kähler form on that space. The basic model is a blowup, where a small exceptional correction repairs the degeneracy of a pulled-back form. Contraction reverses the problem and requires a genuine descent and positivity theorem.

1 The local model: repairing a pullback

Proposition 1.1. *Let $\mu : \widetilde{X} \to X$ be the blowup of a smooth compact Kähler manifold along a smooth centre, with exceptional divisor E. If ω is a Kähler form on X, then $\mu^*[\omega] - \varepsilon[E]$ is Kähler for every sufficiently small positive ε.*

Proof. The bundle $\mathcal{O}_E(-E)$ is the relative $\mathcal{O}(1)$ of the projective normal bundle. Choose a smooth metric on $\mathcal{O}_{\widetilde{X}}(-E)$ whose curvature η is positive on the tangent spaces to the fibres of $E \to \mu(E)$. Such a metric is obtained by taking a fibrewise Fubini–Study metric on E and extending it smoothly. Precisely these vertical tangent spaces form the kernel of the semipositive form $\mu^*\omega$ along E.

We use a compact linear-algebra observation. If a continuous semipositive Hermitian form A on a compact vector bundle has a continuous Hermitian form B positive on every nonzero vector in $\ker A$, then $A + \varepsilon B$ is positive for sufficiently small $\varepsilon > 0$. Otherwise there would be unit vectors v_j and $\varepsilon_j \to 0$ with $A(v_j, v_j) + \varepsilon_j B(v_j, v_j) \le 0$. A convergent subsequence would limit to a unit vector v with $A(v, v) = 0$ and $B(v, v) \le 0$, a contradiction.

Apply this observation on the compact manifold $\widetilde{X}$, with a fixed background Hermitian metric for its unit tangent bundle: away from E the kernel is zero. Thus $\mu^*\omega + \varepsilon\eta$ is positive everywhere. It is closed and represents the stated class. ∎

The sign is forced: subtracting E adds positivity in the directions collapsed by the blowup. This argument is the geometric content of the standard blowup lemma; see [Bou02].

2 The contraction theorem needed in dimension three

Theorem 2.1 (Höring–Peternell, Corollary 8.2)**.** *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities, and let R be a K_X-negative extremal ray of the generalized Mori cone. Suppose $f : X \to Y$ is a bimeromorphic contraction, $-K_X$ is f-ample, and the curves contracted by f are precisely those with class in R. Then Y is Kähler [HP16].*

The strategy is to choose a nef supporting class vanishing on R, descend it to the target, and prove that the descended class is Kähler. Descent is cohomological; strict positivity downstairs uses a Kählerness criterion. The theorem is a supporting analytic input to the existence chapter. Generalized threefold variants and their hypotheses are treated in [DHY23].

For a flip one must also establish Kählerness of the flipped space. Projectivity of the flipping morphism is useful here, but projectivity over a target should not be confused with the target being projective. The MMP chapter states the relevant preservation results together with the step-existence theorems.

3 Why curve positivity alone is not a proof

On a nonprojective compact Kähler manifold, the algebraic curves need not test every real $(1,1)$-direction. A descended class can therefore pass all curve tests without the argument having established Kählerness. The smooth Demailly–Păun criterion instead identifies the Kähler cone as a connected component of the cone of classes with positive top intersection on every positive-dimensional irreducible analytic subvariety [DP04]. The component condition also matters. On singular targets one needs the applicable singular criterion, rather than quoting the smooth statement verbatim.

4 Research directions

The useful separation is between descent of a Bott–Chern class and positivity of its descended representative. The blowup calculation suggests how exceptional corrections can repair positivity upstairs; it does not justify arbitrary blowdowns. Higher-dimensional

extensions must explain how the supporting class is controlled on higher-dimensional subvarieties of the target, and how the singularities permit descent of local potentials.

References

[Bou02] S. Boucksom, *On the volume of a line bundle*, International Journal of Mathematics **13** (2002), no. 10, 1043–1063.

[DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.

[DP04] J.-P. Demailly and M. Păun, *Numerical characterization of the kähler cone of a compact kähler manifold*, Annals of mathematics (2004), 1247–1274.

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.

On the Kähler 3-fold and 4-fold MMP

Kähler MMP • Contractions and flips

Yi Li

Contents

1 Extremal contractions and the trichotomy **2**

2 The analytic toolbox **4**

3 The flip is finite generation **5**

4 The engine: an all-dimensional PLT contraction theorem **7**

5 Das–Hacon I: the divisorial contraction **8**

5.1 Step 1: the null locus is an irreducible Moishezon surface 8

5.2 Step 2: the nef reduction dichotomy 9

5.3 Step 3: a $\mathbb{Q}$-factorial minimal dlt modification 9

5.4 Step 4: the modification is trivial away from the null locus 10

5.5 Step 5: running the α'-trivial MMP 10

5.6 Step 6: controlling what is contracted, and $\Theta^m = 0$ 11

5.7 Step 7: the map descends to a morphism 12

5.8 Step 8: base-point freeness and $\alpha = f^*\alpha_Z$ 13

6 Das–Hacon II: the flipping contraction **13**

6.1 The terminal route 13

6.2 The klt/dlt route 14

7 The Höring–Peternell approach **15**

7.1 Divisorial type, contraction to a point ($n(\alpha) = 0$) 15

7.2 Divisorial type, contraction to a curve ($n(\alpha) = 1$) 15

7.3 Flipping type (terminal 3-folds) 16

8 Comparison of the two approaches 18

Big picture

Once the cone theorem produces a $(K_X + B)$-negative extremal ray R, the next step of the minimal model program is to *contract* it. For a $\mathbb{Q}$-factorial compact Kähler 3-fold klt pair the contraction splits into two cases according to the dimension of the null locus of the nef and big supporting class α: **divisorial type** when $\dim \mathrm{Null}(\alpha) = 2$, and **flipping type** when $\dim \mathrm{Null}(\alpha) \leq 1$. This note treats both, because the two cases are proved by the same two competing methods and share the same engine, and separating them hides that.

Section 1 fixes the definitions, records the structural properties any contraction must have, and states the two theorems everything below establishes. Section 2 collects the analytic facts used throughout. Section 3 disposes of the *flip*: given a flipping contraction, the flip is a relative Proj of a finitely generated algebra, so the work is always the contraction, never the flip. Section 4 isolates the engine both methods run on — an all-dimensional contraction theorem for PLT pairs.

The two methods then appear in full. Sections 5 and 6 give the **Das–Hacon** approach: lift to a $\mathbb{Q}$-factorial dlt model, run an α'-trivial MMP that surgers only the null locus, and show the result descends to a base-point-free morphism. Section 7 gives the earlier **Höring–Peternell** approach, tied to surface and bundle geometry: contract the null locus by hand, using Kleiman plus Grauert, or a $\mathbb{P}^1$-bundle structure plus Nakano. Section 8 compares them.

The references are [DH24] for the Das–Hacon approach, [HP16] and [HP24] for the Höring–Peternell approach, and [DH20, DHYn25] for the log and generalized-pair refinements. Throughout we work with normal compact analytic varieties in Fujiki's class $\mathcal{C}$; (X, B) is a compact Kähler pair, $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is a Bott–Chern class, $\overline{\mathrm{NA}}(X)$ is the generalized Mori cone (the closure of the cone of classes of positive currents), and $\mathrm{Null}(\alpha) = \bigcup_{\int_V (\alpha|_V)^{\dim V} = 0} V$ is the null locus of a nef and big class. A bimeromorphic morphism $f : X \to Z$ is **small** if $\dim \mathrm{Ex}(f) \leq \dim X - 2$, equivalently if f is an isomorphism in codimension 1.

1 Extremal contractions and the trichotomy

Definition 1.1 ([DH24, Def. 3.10.4]). Let $(X, B + \boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial compact Kähler pair. A bimeromorphic morphism $f : X \to Z$ to a normal compact complex space is a **divisorial contraction** if

(1) the relative Picard number is $\rho(X/Z) = 1$ and the exceptional locus $\mathrm{Ex}(f)$ is a divisor, and

(2) $-(K_X + B + \boldsymbol{\beta})$ is f-ample (equivalently, relatively Kähler over Z).

Any f meeting Definition 1.1 inherits the following, exactly as in the projective MMP; these are the properties the existence proofs are engineered to realize.

Proposition 1.2 (Basic properties, [DH24, Thm. 2.5.1]). *Let $f : X \to Z$ be a divisorial contraction of a $(K_X + B)$-negative extremal ray, with X $\mathbb{Q}$-factorial and DLT. Then:*

(1) $\mathrm{Ex}(f) = E$ *is a single* prime *divisor;*

(2) $\rho(X/Z) = 1$, *so* $\rho(Z) = \rho(X) - 1$;

(3) Z *is* $\mathbb{Q}$*-factorial;*

(4) $(Z, B_Z = f_*B)$ *is again DLT (resp. KLT), and one may write* $K_X + B = f^*(K_Z + B_Z) + eE$ *with discrepancy* $e > 0$.

Definition 1.3 (flipping contraction and flip; [DH24, Def. before Thm. 5.12]). Let $(X, B + \boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial generalized log pair. A **flipping contraction** is a small bimeromorphic morphism $f : X \to Z$ with $\rho(X/Z) = 1$ such that $-(K_X + B + \boldsymbol{\beta}_X)$ is Kähler over Z. The **flip** of f (if it exists) is a small bimeromorphic morphism $f^+ : X^+ \to Z$ with $\rho(X^+/Z) = 1$ such that $K_{X^+} + B^+ + \boldsymbol{\beta}_{X^+}$ is Kähler over Z, where B^+ is the strict transform of B. The diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & Z & \end{array}$$

is the **flip diagram**.

⚠ **Warning.** Unlike a divisorial contraction, a small contraction cannot be continued: the target of a nontrivial small $(K_X + B)$-negative contraction is never $\mathbb{Q}$-factorial, and a small contraction onto a $\mathbb{Q}$-factorial variety is an isomorphism. This is the structural reason the MMP must be repaired by *replacing* X with the flip X^+ rather than passing to Z.

Observation 1.4 (trichotomy of extremal contractions). A birational contraction of a $(K_X + B)$-negative extremal ray is either a divisorial contraction (exceptional locus a divisor) or of **flipping type** (exceptional locus of codimension ≥ 2). For a compact Kähler 3-fold this dichotomy is read off the null locus of the supporting class α: the ray is flipping precisely when $\dim \mathrm{Null}(\alpha) \leq 1$.

Observation 1.4 is what organizes this note. The single statement covering both branches is the following; the rest of the note is its proof, case by case.

Theorem 1.5 (contraction theorem for Kähler 3-folds; [DH24, Thm. 1.2]). *Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler 3-fold dlt pair with $K_X + B$ pseudo-effective. Let ω be a Kähler class such that $\alpha = K_X + B + \omega$ is nef and big and $\alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$ is an extremal ray. Then there is a projective bimeromorphic morphism $f : X \to Z$ with connected fibers such that $\alpha = f^*\alpha_Z$ for a Kähler class α_Z on Z. If f is divisorial then (Z, f_*B) is $\mathbb{Q}$-factorial klt; if R is of flipping type then f is a flipping contraction and its flip exists.*

Split along $\dim \mathrm{Null}(\alpha)$, Theorem 1.5 is the conjunction of the next two theorems. Note that the hypotheses are identical except for the last line.

Theorem 1.6 (divisorial type; [DH24, Thm. 6.9]). *Let (X, B) be a strongly $\mathbb{Q}$-factorial compact Kähler 3-fold klt pair such that*

(1) $K_X + B$ is pseudo-effective,

(2) $\alpha = [K_X + B + \beta]$ is nef and big for some Kähler class β, and

*(3) $R = \alpha^\perp \cap \overline{\mathrm{NA}}(X)$ is an extremal ray of **divisorial type**, i.e. $\dim \mathrm{Null}(\alpha) = 2$.*

Then the divisorial contraction $f : X \to Z$ of R exists, and there is a Kähler class α_Z on Z with $\alpha = f^*\alpha_Z$.

Theorem 1.7 (flipping type; [DH24, Thm. 6.2]). *Let (X, B) be a strongly $\mathbb{Q}$-factorial compact Kähler 3-fold klt pair with $K_X + B$ pseudo-effective. Suppose $\alpha = [K_X + B + \beta]$ is nef and big for some Kähler β and $\alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$ is an extremal ray of **flipping type** (i.e. $\dim \mathrm{Null}(\alpha) \leq 1$). Then the flipping contraction $f : X \to Z$ and the flip $X \dashrightarrow X^+$ exist, and there is a Kähler class α_Z on Z with $\alpha = f^*\alpha_Z$.*

Note that since R is α-trivial and β is Kähler, the hypotheses force R to be $(K_X + B)$-negative: $(K_X + B) \cdot R = (\alpha - \beta) \cdot R = -\beta \cdot R < 0$. In both theorems the class α plays a double role — it is the supporting class cutting out R, and the target of a descent $\alpha = f^*\alpha_Z$ that certifies base-point freeness.

2 The analytic toolbox

The obstruction to importing the projective proofs wholesale is that a nef supporting class of an extremal ray is a transcendental Bott–Chern $(1,1)$-class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$, not (a multiple of) a $\mathbb{Q}$-divisor, so the base-point-free theorem is unavailable and semi-positivity replaces semi-ampleness. Both methods below are ways of manufacturing an honest contraction out of such an α, and both rest on the following facts.

Fact 2.1 (nef supporting class; [HP16, Prop. 7.3]). Let X be a $\mathbb{Q}$-factorial compact Kähler 3-fold with K_X pseudo-effective and $R = \mathbb{R}^+[\Gamma_{i_0}]$ a K_X-negative extremal ray of $\overline{\mathrm{NA}}(X)$.

Then there is a nef class $\alpha \in N^1(X)$ with $R = \{z \in \overline{\mathrm{NA}}(X) \mid \alpha \cdot z = 0\}$ which is strictly positive on $\big(\overline{\mathrm{NA}}(X)_{K_X \geq 0} + \sum_{i \neq i_0} \mathbb{R}^+[\Gamma_i]\big) \setminus \{0\}$. We call α a **nef supporting class** of R.

Fact 2.2 (Kleiman criterion, Kähler/Fujiki 3-folds)**.** On a compact Kähler (Fujiki-class) 3-fold X, a class $\gamma \in H^{1,1}_{\mathrm{BC}}(X)$ is Kähler if and only if $\gamma \cdot z > 0$ for every $z \in \overline{\mathrm{NA}}(X) \setminus \{0\}$.

Fact 2.3 (null locus = non-Kähler locus; Nakamaye)**.** For a nef and big class α on a compact Kähler manifold, the non-Kähler locus $E_{nK}(\alpha)$ equals the null locus $\bigcup_{\alpha^{\dim Z} \cdot Z = 0} Z$. A big class whose non-Kähler locus contains no divisor is **modified Kähler**; it then admits a modification $\mu\colon \widetilde{X} \to X$ with $\mu^*\alpha = \widetilde{\alpha} + E$, where $\widetilde{\alpha}$ is Kähler, $E \geq 0$ is μ-exceptional and $\mu(\mathrm{Supp}\, E) = E_{nK}(\alpha)$.

Fact 2.4 (Grauert–Fujiki contractibility)**.** Let $\Xi = \sum m_i D_i$ be a compact analytic divisor in a normal variety $\widehat{X}$ whose conormal bundle $N^*_{\Xi/\widehat{X}} = \mathcal{O}_\Xi(-\Xi)$ is ample. Then there is a bimeromorphic morphism $\psi\colon \widehat{X} \to Y$ contracting Ξ to a point and isomorphic elsewhere. Any contraction is moreover determined by the curves it contracts (rigidity lemma).

Fact 2.5 (finite generation over Fujiki-class klt bases; [DHP24, Thm. 1.3], [Fuj22, Thm. 1.22])**.** Let $f : X \to Y$ be a proper surjective morphism with X in Fujiki's class $\mathcal{C}$ and (X, B) klt. Then $R(X/Y, K_X + B) = \bigoplus_{m \geq 0} f_*\mathcal{O}_X(m(K_X + B))$ is locally finitely generated.

3 The flip is finite generation

In the projective MMP the existence of flips is the hard analytic core, settled by finite generation of the log canonical algebra (BCHM). Passing to the compact Kähler / Fujiki-class setting *redistributes* this difficulty. The Kähler notion of a flip does not require the contraction to be projective, so once a flipping *contraction* is available, producing the flip is comparatively soft. We dispose of that half first, so that the rest of the note can concentrate on the contraction.

Theorem 3.1 (Kollár–Mori; Birkar)**.** *Let $f : X \to Z$ be a small proper morphism of normal varieties and D a $\mathbb{Q}$-Cartier divisor which is f-anti-ample. Then the D-flip diagram exists if and only if the relative section algebra*

$$\mathcal{R}(D/Z) \;=\; \bigoplus_{m \geq 0} f_*\,\mathcal{O}_X\big(\lfloor mD \rfloor\big)$$

is a locally finitely generated $\mathcal{O}_Z$-algebra. When it exists the flip is

$$X^+ = \operatorname{Proj}_Z \mathcal{R}(D/Z) \longrightarrow Z,$$

and it is unique.

Idea of proof. The flip, if it exists, *is* the relative Proj of the section algebra; so the only issue is whether that Proj is of finite type over Z (hence a variety). One direction is the tautology "Proj of a finitely generated algebra"; the reverse direction observes that $X \dashrightarrow X^+$ is an isomorphism in codimension 1, so Hartogs identifies the two section algebras and the flipped divisor D^+ is f^+-ample — and an ample divisor's section algebra is finitely generated.

Proof. Finite generation is local on Z, so we may take $Z = \operatorname{Spec} A$ (resp. a Stein base) and $R := \mathcal{R}(D/Z)(Z)$ finitely generated over A.
($\Leftarrow$) After truncating we may assume R is generated in degree 1; replacing D by a multiple (which does not change the flip) set $X^+ := \operatorname{Proj} R$ with $f^+ \colon X^+ \to Z$ and $\mathcal{O}_{X^+}(1)$ f^+-ample. For $m \gg 0$ one recovers $f_*\mathcal{O}_X(mD) = f^+_*\mathcal{O}_{X^+}(m)$, so X^+ is normal. Over the smooth locus $V \subset Z$ both f and f^+ are isomorphisms (any curve contracted by a flipping contraction lies over the singular locus), so f^+ is bimeromorphic. Finally f^+ is small: if some divisor $E \subset X^+$ were f^+-contracted, then for the divisor L with $\mathcal{O}_{X^+}(1) = \mathcal{O}_{X^+}(L)$ the inclusion $\mathcal{O}_{X^+}(mL) \subsetneq \mathcal{O}_{X^+}(mL + E)$ stays strict after f^+_*, because $R^1 f^+_* \mathcal{O}_{X^+}(mL) = 0$ by relative Serre vanishing — contradicting $f^+_*\mathcal{O}_{X^+}(mL) = f_*\mathcal{O}_X(mD)$.
($\Rightarrow$) Given the flip $f^+ \colon X^+ \to Z$, the birational map $X \dashrightarrow X^+$ is an isomorphism in codimension 1. On a common open $V \cong V'$ we have $\mathcal{O}(D)|_V \cong \mathcal{O}(D^+)|_{V'}$, and by Hartogs' extension (both spaces are normal, hence S_2) $H^0(f^{-1}U, \mathcal{O}(D)) \cong H^0((f^+)^{-1}U, \mathcal{O}(D^+))$ for all $U \subset Z$; thus $\mathcal{R}(D/Z) \cong \mathcal{R}(D^+/Z)$. But D^+ is f^+-ample, so its relative section algebra is finitely generated. Uniqueness is immediate from $X^+ = \operatorname{Proj}_Z \mathcal{R}(D/Z)$. ∎

Theorem 3.2 (existence of the flip; [DH24, Thm. 5.12])**.** *Let $(X, B + \beta)$ be a $\mathbb{Q}$-factorial gdlt pair with X in Fujiki's class $\mathcal{C}$ and $f \colon X \to Z$ a flipping contraction (Definition 1.3). Then the flip f^+ of f exists; it is locally projective over Z.*

Proof sketch. The flipping contraction is (locally) a projective morphism, and $(X, B + \beta)$ is klt away from the boundary. By Fact 2.5 the relative log canonical algebra is locally finitely generated over Z; Theorem 3.1 then produces $X^+ = \operatorname{Proj}_Z \mathcal{R}$ locally, and these local flips glue to the global $f^+ \colon X^+ \to Z$. ∎

In the Kähler MMP the flip follows from the contraction by finite generation (Theorem 3.1+Fact 2.5); the theorem to prove is the existence of the contraction.

4 The engine: an all-dimensional PLT contraction theorem

Both branches of the Das–Hacon proof, divisorial and flipping, are driven by one theorem. It converts a contraction *on a boundary divisor* into a contraction of the ambient space, and it is the only place where the analytic hard work happens.

Theorem 4.1 ([DH24, Thm. 5.8])**.** *Let $(X, S + B + \boldsymbol{\beta})$ be a generalized PLT pair with X a compact analytic variety and S $\mathbb{Q}$-Cartier. Suppose*

(1) $\lfloor S + B \rfloor = S$ is irreducible,

(2) there is a proper morphism $\pi\colon S \to T$ with $\pi_* \mathcal{O}_S = \mathcal{O}_T$,

(3) $-(K_X + S + B + \boldsymbol{\beta}_X)|_S$ is relatively Kähler over T, and

(4) $-S|_S$ is π-ample.

Then there is a bimeromorphic morphism $p\colon X \to Z$ with $p|_S = \pi$ and $p|_{X \setminus S}$ an isomorphism. If p is of flipping type, the flip exists.

Remark 4.2. The non-generalized case of Theorem 4.1 — $(X, S + B)$ a $\mathbb{Q}$-factorial compact Kähler plt pair, $\pi\colon S \to T$ a contraction with $-(K_X + S + B)|_S$ and $-S|_S$ both π-ample — is stated separately as [DH24, Thm. 1.1], and yields in addition that p is projective with connected fibers. The two are used interchangeably below.

Idea of proof. Apply Fujiki's blow-down theorem, but not to S directly — S is only $\mathbb{Q}$-Cartier. Instead blow down a Cartier thickening $A = \ell S$. Two things must be arranged: that $-A|_A$ stays π_A-ample on the thickening (an "ample on the reduction $\Rightarrow$ ample" argument), and, the technical heart, the vanishing $R^1\pi_* \mathcal{E}_k = 0$ of the first direct image of the graded pieces $\mathcal{E}_k = \mathcal{O}_X(-kS)/\mathcal{O}_X(-(k+1)S)$. This vanishing is what lets the ring structure of $\bigoplus \pi_* \mathcal{O}_{S_k}$ integrate the infinitesimal fibrations $\pi_k\colon S_k \to T_k$ into an honest contraction. It follows from Kawamata–Viehweg vanishing applied to the GKLT pair extracted from $(X, S + B + \boldsymbol{\beta})$ by adjunction, since $-(K_X + S + B + \boldsymbol{\beta}_X)|_S$ is relatively big and nef.

Sketch. By adjunction S is normal, $(S, B_S + \boldsymbol{\beta}_S)$ is GKLT and $-(K_S + B_S + \boldsymbol{\beta}_S) = -(K_X + S + B + \boldsymbol{\beta}_X)|_S$ is Kähler over T. Choosing $\mathcal{L} = \mathcal{O}_X(-K_X - (k+1)S)$ in the adjunction sequence yields

$$0 \to \mathcal{O}_X(-(k+1)S) \to \mathcal{O}_X(-kS) \to \mathcal{E}_k \to 0,$$

with $\mathcal{E}_k$ a rank-1 reflexive sheaf on S; writing S_k for the thickening $\mathcal{O}_{S_k} = \mathcal{O}_X/\mathcal{O}_X(-kS)$ gives the ring sequence $0 \to \mathcal{E}_k \to \mathcal{O}_{S_{k+1}} \to \mathcal{O}_{S_k} \to 0$.

Vanishing. Locally over T one has $\mathcal{E}_k = \mathcal{O}_S(G_k)$ with

$$G_k \sim_{\mathbb{Q}} K_S + (B_S - \Delta_k) + \boldsymbol{\beta}_S - (K_S + B_S + \boldsymbol{\beta}_S) - kS|_S, \qquad 0 \le \Delta_k \le B_S,$$

so $(S, (B_S - \Delta_k) + \boldsymbol{\beta}_S)$ is GKLT and the twist $-(K_S + B_S + \boldsymbol{\beta}_S) - kS|_S$ is relatively Kähler; Kawamata–Viehweg vanishing gives $R^1\pi_*\mathcal{E}_k = 0$ (when B_S is not $\mathbb{Q}$-Cartier one first passes to a small $\mathbb{Q}$-factorialization $q\colon \tilde{S} \to S$, applies vanishing there, and descends by a spectral-sequence argument). Hence $\pi_*\mathcal{O}_{S_{k+1}} \to \pi_*\mathcal{O}_{S_k}$ is surjective, so $T_k := \mathrm{Specan}(\pi_*\mathcal{O}_{S_k})$ are compatible thickenings of T and the $\pi_k\colon S_k \to T_k$ are morphisms of non-reduced spaces. *Blow-down.* Fix ℓ with $A = \ell S$ Cartier. Since $-A|_A$ pulls back to the π-ample $-\ell S|_S$ on the reduction and the induced map on bases is finite, $-A|_A$ is π_A-ample; a change-of-index plus Serre vanishing argument upgrades this to $R^i\pi_{A,*}\mathcal{O}_A(-kA) = 0$ for $i, k \geq 1$. Fujiki's blow-down theorem then produces $p\colon X \to Z$ with $p|_A = \pi_A$ and $p|_{X\setminus A}$ an isomorphism; restricting, $p|_S = \pi$. Flips of flipping type exist by analytic BCHM [DH24]. ∎

Remark 4.3. The Cartier hypothesis on S is exactly what would let one quote KV vanishing directly; its failure is absorbed into the "variation of the different" correction term Δ_k, whose GKLT-ness is what the proof buys. This is why the theorem holds *in all dimensions* and is the load-bearing input to the 3-fold constructions below.

5 Das–Hacon I: the divisorial contraction

The strategy is dimension-independent in its engine and surgical in its execution. The engine is Theorem 4.1. Its input — a divisor S carrying a fibration with the right relative positivity — is produced by localizing on the null locus (Section 5.1), passing to a dlt model (Section 5.3), and running an MMP that touches only that locus (Sections 5.5 and 5.6). The output is shown to be a genuine base-point-free morphism (Sections 5.7 and 5.8). This section proves Theorem 1.6.

5.1 Step 1: the null locus is an irreducible Moishezon surface

Proposition 5.1 ([DH24, Prop. 6.10])**.** *Under the hypotheses of Theorem 1.6 there is an irreducible divisor S with $\mathrm{Null}(\alpha) = S$; moreover S is a Moishezon surface covered by a family of α-trivial curves, and $S \cdot R < 0$. In particular $R = \mathbb{R}^+[C_t]$ for the curves C_t of this family.*

Idea of proof. Perturb α by a small Kähler class and take the Boucksom–Zariski decomposition $\alpha - \epsilon\omega = \sum s_j S_j + P$. A component $S = S_1$ of the negative part carries $\alpha|_{S'}$ nef but not big; feeding the decomposition through adjunction shows $K_{S'}$ is not pseudo-effective, so the minimal resolution S' is uniruled, hence projective and Moishezon. A movable, α-trivial curve on S' is produced by Batyrev's decomposition of the nef curve class $\alpha|_{S'}$; being movable it covers S.

Proof. Let ω be Kähler. As α is big, $\alpha - \epsilon\omega$ is big for $0 < \epsilon \ll 1$, with Boucksom–Zariski decomposition $\alpha - \epsilon\omega = \sum s_j S_j + P$, P modified nef and big [Bou04]. If $S \subset \mathrm{Null}(\alpha)$ is

a surface and $S' \to S$ its minimal resolution, then $\alpha|_{S'}$ is nef with $(\alpha|_{S'})^2 = 0$, so S is a component of the negative part, say $S = S_1$.
Put $b = \mathrm{mult}_S(B)$. Since $\mathrm{mult}_S\big(B + \frac{1-b}{s_1}\sum s_j S_j\big) = 1$, adjunction on S' gives

$$\Big(1 + \tfrac{1-b}{s_1}\Big)\alpha\Big|_{S'} = K_{S'} + B_{S'} + \Big(\beta + \tfrac{1-b}{s_1}(P + \epsilon\omega)\Big)\Big|_{S'}, \qquad B_{S'} \geq 0.$$

The bracketed class is big (Kähler + pseudo-effective) while $\alpha|_{S'}$ is not big, so $K_{S'}$ is not pseudo-effective. Hence S' is a projective uniruled surface with $H^2(S', \mathcal{O}_{S'}) = 0$, and S is Moishezon.
Because $H^2(S', \mathcal{O}_{S'}) = 0$, the nef class $\alpha|_{S'}$ is represented by an $\mathbb{R}$-divisor, and as a nef curve it lies in $\overline{\mathrm{NM}}_1(S')$. Batyrev's theorem [HP16] gives $\alpha|_{S'} \equiv C_\epsilon + \sum_i \lambda_i M_i$ with M_i movable. If some $\alpha|_{S'} \cdot M_i = 0$, then M_i pushes to a movable α-trivial curve on S and we are done. Otherwise $\alpha|_{S'} \cdot M_i > 0$ for all i; then $0 = (\alpha|_{S'})^2 = \alpha|_{S'} \cdot (C_\epsilon + \sum \lambda_i M_i)$ forces $k = 0$ and $\alpha|_{S'} \equiv C_\epsilon$, whence $\alpha|_{S'} \cdot (K_{S'} + \epsilon A_{S'}) \geq 0$ and, letting $\epsilon \to 0$, $\alpha|_{S'} \cdot K_{S'} \geq 0$. Combined with the displayed adjunction identity and $(\alpha|_{S'})^2 = 0$, every term is forced to vanish; in particular $\alpha|_{S^\nu} \cdot \omega|_{S^\nu} = 0$ on the normalization, so $\alpha|_{S^\nu} \equiv 0$ — contradicting $\alpha|_{S'} \cdot M_i > 0$. Thus the covering family exists.
Finally, for a covering α-trivial curve C_t,

$$0 > -\epsilon\omega \cdot C_t = (\alpha - \epsilon\omega) \cdot C_t = \Big(\sum s_j S_j + P\Big) \cdot C_t \geq s_1\, S \cdot C_t,$$

using $S_j \cdot C_t \geq 0$ for $j \neq 1$. Hence $S \cdot C_t < 0$, so $S \cdot R < 0$. The same computation applied to any curve C with $[C] \in R$ gives $S \cdot C < 0$, forcing $S = \mathrm{Null}(\alpha)$ with no other component. ∎

5.2 Step 2: the nef reduction dichotomy

Let $S^\nu \to S$ be the normalization and $n(\alpha) := n(\alpha|_{S^\nu})$ the nef dimension of $\alpha|_{S^\nu}$. Since S is covered by α-trivial curves, $\alpha|_{S^\nu}$ is not big, so $0 \leq n(\alpha) \leq 1$. The nef reduction [HP16] is a morphism $\varphi\colon S^\nu \to T$ with:

- $n(\alpha) = 0$: $\alpha|_{S^\nu} \equiv 0$ and T is a point;
- $n(\alpha) = 1$: T is a smooth curve, $\alpha|_{S^\nu} = \varphi^*\gamma$, and a curve C is α-trivial iff it is vertical over T.

The whole construction below runs uniformly in $n(\alpha)$, the two cases differing only in which components of μ^*S end up contracted.

5.3 Step 3: a $\mathbb{Q}$-factorial minimal dlt modification

Put $b = \mathrm{mult}_S(B)$ and $\Delta = B + (1-b)S$, so $\lfloor\Delta\rfloor \supseteq S$ and $\alpha = [K_X + B + \beta]$.

Proposition 5.2 ([DH24, Claim 6.11]). *There is a projective bimeromorphic $\mu\colon X' \to X$ such that, with $\Delta' = \mu_*^{-1}\Delta + \mathrm{Ex}(\mu)$:*

(1) (X', Δ') is $\mathbb{Q}$-factorial and DLT;

(2) $K_{X'} + \Delta'$ is nef over X, so $\mu^*(K_X + \Delta) - (K_{X'} + \Delta') \geq 0$;

(3) $K_{X'} + \Delta' = \mu^*(K_X + B) + (1-b)S' + \sum a_j E_j$ with $S' = \mu_*^{-1}S$ and all $a_j > 0$.

Proof. Take a projective log resolution $\mu\colon X' \to X$ with $\Delta' = \mu_*^{-1}\Delta + \mathrm{Ex}(\mu)$; since $\mathrm{Ex}(\mu)$ is SNC, (X', Δ') is log smooth, hence DLT. Run the $(K_{X'} + \Delta')$-MMP over X (analytic BCHM [DH24]); it is automatically of general type over X as μ is bimeromorphic, and it preserves DLT and $\mathbb{Q}$-factoriality, giving (1). By termination we may assume $K_{X'} + \Delta'$ is nef over X; the negativity lemma then yields $\mu^*(K_X + \Delta) - (K_{X'} + \Delta') \geq 0$, which is (2). For (3): as (X, B) is klt, $\mu^*(K_X + B) = K_{X'} + \tilde{B} + \sum b_j E_j$ with $b_j < 1$; comparing with $\Delta' = (1-b)S' + \tilde{B} + \mathrm{Ex}(\mu)$ gives $\mathrm{Ex}(\mu) = \sum(b_j + a_j)E_j$, so $a_j = 1 - b_j > 0$. ∎

Condition (2) is the technical payload — it provides the relative nef/positivity later fed to the negativity lemma — while (3) records that the locus we intend to contract (S' and $\mathrm{Ex}(\mu)$) is retained with positive coefficient, hence sits inside $\lfloor \Delta' \rfloor$.

5.4 Step 4: the modification is trivial away from the null locus

Write $U = X \setminus S = X \setminus \mathrm{Null}(\alpha)$, $X'_U = \mu^{-1}(U)$, $\alpha' = \mu^*\alpha$.

Lemma 5.3 ([DH24, p. 36]). *(0) $(X, \Delta|_U) = (U, B|_U)$ is klt;*

(1) $\mathrm{Ex}(\mu)$ is divisorial and $\mu|_{X'_U}\colon X'_U \to U$ is an isomorphism;

(2) if a curve $C \not\subset X' \setminus X'_U$, then $\alpha' \cdot C > 0$;

(3) every α'-trivial step of the $(K_{X'} + \Delta')$-MMP involves only curves in $X' \setminus X'_U$.

Proof. On U, $\Delta|_U = B|_U$ since $S = \mathrm{Null}(\alpha)$ is removed, so $(U, \Delta|_U)$ is klt. As X is $\mathbb{Q}$-factorial and X' has rational (DLT) singularities, $\mathrm{Ex}(\mu)$ is a divisor, so $\mu|_{X'_U}$ is an isomorphism. For (2): $\mu^*\alpha \cdot C = \alpha \cdot \mu_* C \geq 0$ by nefness, and if it were 0 then $\mu(C) \subset \mathrm{Null}(\alpha) = X \setminus U$, forcing $C \subset X' \setminus X'_U$; contrapositive is (2). Then (3) is immediate, since α'-trivial contracted curves have $\alpha' \cdot C = 0$ and hence lie in $X' \setminus X'_U$. ∎

5.5 Step 5: running the α'-trivial MMP

Proposition 5.4 ([DH24, Claim 6.12]). *One can run an α'-trivial $(K_{X'} + \Delta')$-MMP $X' = X^0 \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow X^m$ with $\phi^i\colon X' \dashrightarrow X^i$ such that:*

(1) each step is α'-trivial, so $\alpha^i = \phi^i_* \alpha'$ is nef;

(2) $\phi^i|_{X'_U}$ is an isomorphism onto X^i_U;

(3) $(X^i, \Delta^i = \phi^i_*\Delta')$ *is DLT,* X^i *strongly* $\mathbb{Q}$*-factorial, with* $\lfloor\Delta^i\rfloor \subset W^i := X^i \setminus X^i_U$;
(4) $K_{X^i} + \Delta^i - \Theta^i \equiv_{\alpha^i} 0$ *with* $\mathrm{Supp}(\Theta^i) = \lfloor\Delta^i\rfloor$;
(5) *if* S^m *is a component of* $\lfloor\Delta^m\rfloor$*, then every* $(K_{S^m} + \Delta_{S^m})$*-negative extremal ray* R^m *is* α_{S^m}*-positive,* $\alpha_{S^m} \cdot R^m > 0$.

Idea of proof. An ordinary MMP need not exist here; the point is to run it α'*-trivially*, so that the only contractions performed are the ones supported on $\lfloor\Delta'\rfloor = \mu_*^{-1}S \cup \mathrm{Ex}(\mu)$. Each contraction is manufactured by finding a $(K_{S^m} + \Delta_{S^m})$-negative, α-trivial extremal ray *on a boundary surface* $\bar{S}^i$, contracting it there, and extending the surface contraction to X^i by the PLT theorem Theorem 4.1. Termination is the 3-fold termination of [DH24]; condition (5) is the stopping criterion, signalling nothing α-trivial is left to contract on the NKLT locus.

Proof sketch. *Base case.* $\alpha' = \mu^*\alpha$ is nef. Since $(K_X + B) \cdot R < 0$ and $S \cdot R < 0$, there is $s > 0$ with $(K_X + B - sS) \cdot R = 0$, i.e. $K_X + B - sS \equiv_\alpha 0$; pulling back,

$$K_{X'} + \Delta' \equiv_{\alpha'} \Theta' := (1-b)S' + \sum a_j E_j + s\mu^*S \geq 0, \qquad \mathrm{Supp}(\Theta') = \lfloor\Delta'\rfloor.$$

Inductive step. Suppose a component $S^i \subset \lfloor\Delta^i\rfloor$ carries a $(K_{S^i} + \Delta_{S^i})$-negative, α_{S^i}-trivial extremal ray $R^i = \mathbb{R}^+[\Sigma^i]$ (otherwise (5) holds and we stop). From (4), $\Theta^i \cdot \Sigma^i = (K_{X^i} + \Delta^i) \cdot \Sigma^i < 0$, so some component $\bar{S}^i$ of $\lfloor\Delta^i\rfloor$ has $\bar{S}^i \cdot \Sigma^i < 0$ and $\Sigma^i \subset \bar{S}^i$. On the surface $\bar{S}^i$, Σ^i lies in a $(K_{\bar{S}^i} + \Delta_{\bar{S}^i})$-negative, $\alpha_{\bar{S}^i}$-trivial extremal face; a spanning ray $\bar{\Sigma}^i$ has $\bar{S}^i \cdot \bar{\Sigma}^i < 0$, and its extremal contraction $\pi\colon \bar{S}^i \to V$ satisfies: $-\bar{S}^i$ is π-ample and $-(K_{\bar{S}^i} + \Delta_{\bar{S}^i})$ is relatively Kähler (Picard number 1 over V). The pair $(X^i, \bar{S}^i + B^i)$ is PLT, so Theorem 4.1 supplies a contraction $p\colon X^i \to Z^i$ with $p|_{\bar{S}^i} = \pi$; if p is flipping the flip exists. Set $X^i \dashrightarrow X^{i+1}$. Termination [DH24] yields X^m with (5). Conditions (1)–(4) are preserved: α^i-triviality of p and injectivity of pullback of Bott–Chern classes under Fano-type contractions give $\alpha^i \equiv p^*\alpha_{Z^i}$ with α_{Z^i} nef, and $K_{X^i} + \Delta^i - \Theta^i \equiv_{\alpha^i} 0$ pushes/pulls through each step. ∎

5.6 Step 6: controlling what is contracted, and $\Theta^m = 0$

The MMP $\phi^m\colon X' \dashrightarrow X^m$ is now shown to contract exactly μ^*S, so that downstairs $\phi\colon X \dashrightarrow X^m$ contracts precisely S.

Proposition 5.5 ([DH24, Claims 6.13–6.14])**.** (1) *If* $n(\alpha) = 0$*, then* ϕ^m *contracts* μ^*S*; hence* $\phi\colon X \dashrightarrow X^m$ *contracts* S *and no other divisor.*
(2) *If* $n(\alpha) = 1$*, then* ϕ^m *contracts* S' *and every component of* μ^*S *dominating* T*; hence* ϕ *contracts* S *and may extract only divisors over* S *not dominating* T.
In all cases $\Theta^m = 0$.

Idea of proof. Choose $F' \geq 0$ μ-exceptional with $-F'$ μ-ample and set $\Xi' = \mu^* S + \epsilon F'$. Two facts drive the proof. First, $-(\Xi' - t\alpha')$ restricts to a Kähler (ample over T) class on S' and on each E_j — because $-S|_{S^\nu} \equiv_\alpha \frac{1}{s}\beta|_{S^\nu}$ is relatively ample and α is nef. Second, the negativity lemma gives $p^*\Xi' - q^*\Xi^m \geq 0$ on a common resolution. If a component of $\mu^* S$ survived, a covering family C_t^m of α^m-trivial curves on it would satisfy, using (5) and $K_{X^m} + \Delta^m \equiv_{\alpha^m} \Theta^m$,

$$0 \geq (\Theta^m - \lambda\Xi^m) \cdot C_t^m = (K_{X^m} + \Delta^m - \lambda\Xi^m) \cdot C_t^m \geq -\lambda\Xi^m \cdot C_t^m \geq -\lambda\Xi' \cdot C_t > 0,$$

a contradiction. Hence every intended component is contracted, forcing $\Theta^m = 0$.

Proof that $\Theta^m = 0$. It suffices to show $q^*\Theta^m = 0$ on the common resolution $p\colon W \to X$, $q\colon W \to X^m$, since pullback of Bott–Chern classes under the contraction μ is injective. As $\Theta^m \geq 0$, by the negativity lemma we need $q^*\Theta^m$ effective, exceptional and nef over X. Exceptionality: by Proposition 5.5 both S' and $\mathrm{Ex}(\mu)$ are contracted, so $\mathrm{Supp}(\Theta^m) = \lfloor \Delta^m \rfloor$ meets no strict transform surviving over X. Nefness: take $C \subset W$ with $\mu_* p_* C = 0$. If $q_* C = 0$ then $q^*\Theta^m \cdot C = 0$. If $q_* C \neq 0$ then $\alpha^m \cdot q_* C = \alpha \cdot \mu_* p_* C = 0$, so $q_* C$ is α^m-trivial; if it lies in a component P of Θ^m, then by (5) $(K_{X^m} + \Delta^m)|_P$ is ≥ 0 on α^m-trivial curves, whence

$$q^*\Theta^m \cdot C = (K_{X^m} + \Delta^m) \cdot q_* C \geq 0.$$

Thus $q^*\Theta^m$ is nef over X, and $\Theta^m = 0$. ∎

5.7 Step 7: the map descends to a morphism

Proposition 5.6 ([DH24]). *The bimeromorphic map $\phi\colon X \dashrightarrow X^m$ is a morphism $f\colon X \to Z := X^m$.*

Proof. Let $p\colon W \to X$, $q\colon W \to X^m$ resolve ϕ. By the rigidity lemma it suffices that every p-exceptional curve is q-exceptional. Pick a Kähler class ω^m on X^m. With $E_1, \ldots, E_r$ a basis of $H^{1,1}_{\mathrm{BC}}(W)/p^*H^{1,1}_{\mathrm{BC}}(X)$, write $p^*\omega \equiv q^*\omega^m + \sum e_i E_i$, and choose r with $(\omega + rS) \cdot R = 0$ (possible since $S \cdot R < 0$); then

$$p^*(\omega + rS) \equiv q^*\omega^m + Q, \qquad Q = \sum e_i E_i + r p^* S.$$

For C q-exceptional: if $p_* C = 0$ then $(q^*\omega^m + Q) \cdot C = (\omega + rS) \cdot p_* C = 0$; if $p_* C \neq 0$, then $p^*\alpha \cdot C = q^*\alpha^m \cdot C = 0$ shows $p_* C \in R$, so again $(q^*\omega^m + Q) \cdot C = (\omega + rS) \cdot p_* C = 0$. Thus Q is q-trivial. By Proposition 5.5 ($\Theta^m = 0$) ϕ contracts S and extracts no divisor, so each E_i and p^*S is q-exceptional, hence Q is q-exceptional. The negativity lemma forces $Q = 0$, i.e. $p^*(\omega + rS) \equiv q^*\omega^m$. If some p-exceptional C were not q-exceptional,

$$0 = C \cdot p^*(\omega + rS) = C \cdot q^*\omega^m = q_* C \cdot \omega^m > 0,$$

absurd. So $\phi = f$ is a morphism. ∎

5.8 Step 8: base-point freeness and $\alpha = f^*\alpha_Z$

Proposition 5.7 ([DH24]). *$-(K_X + B)$ is f-ample, and there is a Kähler class α_Z on Z with $\alpha = f^*\alpha_Z$. This completes Theorem 1.6.*

Proof. Since μ and ϕ^m are isomorphisms over U, so is f, and f contracts exactly the α-trivial curves, i.e. the curves in R. From $\alpha = [K_X + B + \beta]$ and $\alpha \equiv_Z 0$ we get $-(K_X + B) \equiv_Z \beta$, which is f-ample. Hence Z has rational singularities, and injectivity of f^* on Bott–Chern classes gives $\rho(X/Z) = 1$ and $\alpha \equiv f^*\alpha_Z$ for some α_Z.
It remains to see α_Z is Kähler; by Demailly–Păun [DP04] it suffices that $\alpha_Z^{\dim W} \cdot W > 0$ for every positive-dimensional $W \subset Z$. If $W \not\subset f(S)$, its strict transform $W' \not\subset S = \mathrm{Null}(\alpha)$, so $\alpha_Z^{\dim W} \cdot W = \alpha^{\dim W'} \cdot W' > 0$. If $W \subset f(S)$, then since R is divisorial $W = f(S)$ is a curve; S is projective (it is f^{-1} of a projective curve), so there is a curve $W' \subset S$ with $f(W') = W$ and $W' \notin R$, giving $\alpha \cdot W' > 0$ and $\alpha_Z \cdot W = \alpha \cdot W' > 0$ by the projection formula. Thus α_Z is Kähler. ∎

6 Das–Hacon II: the flipping contraction

When $\dim \mathrm{Null}(\alpha) \le 1$ the same engine is used, but the surgery is organized differently: there is no null divisor to localize on, so the boundary surfaces of a dlt modification take over that role. We record the short terminal route first, then the substantive klt route, which proves Theorem 1.7.

6.1 The terminal route

Fact 6.1 (contraction from a big nef class with 1-dimensional null locus; [DH24, Prop. 4.5]). Let X be a normal compact Kähler variety with isolated rational singularities and α a nef and big $(1,1)$-class. If $\mathrm{Null}(\alpha)$ is a pure 1-dimensional analytic subset, then there is a proper bimeromorphic morphism $f : X \to Y$ with $\mathrm{Ex}(f) = \mathrm{Null}(\alpha)$ contracting $\mathrm{Null}(\alpha)$ to finitely many points.

Corollary 6.2 (terminal flipping contraction; [DH24, Cor. 4.6]). *Let (X, B) be a compact Kähler 3-fold terminal pair, ω a Kähler class and $\alpha = K_X + B + \omega$ nef but not Kähler with $\mathrm{Null}(\alpha)$ pure 1-dimensional and $\overline{\mathrm{NA}}(X) \cap \alpha^{\perp} = R$ an extremal ray. Then the flipping contraction $f : X \to Z$ and its flip $f^+ : X^+ \to Z$ both exist.*

Proof. A terminal 3-fold has isolated rational singularities, and α is big because its null locus is 1-dimensional (so the non-Kähler locus contains no divisor). By Fact 6.1 there is a small contraction $f : X \to Z$ with $\mathrm{Ex}(f) = \mathrm{Null}(\alpha)$; being $(K_X + B)$-negative and small, it is a flipping contraction. The flip is then obtained from Fact 2.5+Theorem 3.1: construct

the local log canonical models $\operatorname{Proj}_Z \mathcal{R}$ over Z and glue them to f^+. ∎

6.2 The klt/dlt route

Idea of proof. Resolve to a dlt modification $\nu\colon X' \to X$ and detect, on each boundary surface $S' \subset \lfloor \Delta' \rfloor$, a negative extremal ray via a positive *nef threshold* $\tau_{S'}$ (surface cone theorem + adjunction). Extending the resulting surface contraction to X' by the PLT contraction theorem (Theorem 4.1) gives one step of an auxiliary MMP; running it produces a bimeromorphic $\phi\colon X' \dashrightarrow X^+$. The point is to control this MMP so that it is *proper over* $U = X \setminus \operatorname{Null}(\alpha)$ and contracts *exactly* the ν-exceptional divisors — then it descends to a small map $\psi\colon X \dashrightarrow X^+$. A final Grauert contraction of a common-resolution difference class builds $X^+ \to Z$ and the induced f, f^+, after which one checks the numerical properties.

Architecture of the proof of Theorem 1.7. *Step 0 (dlt modification).* Take $\nu\colon X' \to X$ a $\mathbb{Q}$-factorial dlt modification with $\nu^*\alpha = \omega' + G'$, $G' \geq 0$ ν-exceptional and ω' a modified Kähler class, arranged so that $(X', \Delta' + t\omega')$ is gdlt and $K_{X'} + \Delta' + t\omega' + a\alpha'$ is nef for fixed $t, a > 0$, with $\operatorname{Supp} G' = \operatorname{Ex}(\nu) = \lfloor \Delta' \rfloor$.
Step 1 (find the surface contraction). For a component $S' \subset \lfloor \Delta' \rfloor$, adjunction gives $K_{S'} + \Delta_{S'} + t\omega'_{S'} = (K_{X'} + \Delta' + t\omega')|_{S'}$ with $(S', \Delta_{S'})$ dlt and $\omega'_{S'}$ big. Define the nef threshold

$$\tau_{S'} := \inf\{s \geq 0 \mid K_{S'} + \Delta_{S'} + a\alpha_{S'} + s\omega_{S'} \text{ is nef for some } a > 0\}.$$

By the cone theorem for generalized Kähler DLT surfaces, if $\tau_{S'} > 0$ there is a $(K_{S'} + \Delta_{S'})$-negative extremal ray $R_{S'}$ with $(K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'}) \cdot R_{S'} = \alpha_{S'} \cdot R_{S'} = 0$. Put $\tau = \max_{S'} \tau_{S'}$; if $\tau > 0$ then $K_{X'} + \Delta' + \tau\omega' + a\alpha'$ is nef for some $a > 0$ and a component S' carries an extremal ray R_i with $(K_{X'} + \Delta' + \tau\omega') \cdot R_i = \alpha' \cdot R_i = 0$.
Step 2 (MMP step). By the PLT contraction theorem (Theorem 4.1) the surface contraction $\pi\colon S' \to T$ extends to a bimeromorphic $p\colon X' \to Z'$ with $p|_{S'} = \pi$ and $p|_{X' \setminus S'}$ an isomorphism; p is divisorial or flipping, and in the flipping case its flip exists. Iterating yields a $(K_{X'} + \Delta' + t\omega' + a\alpha')$-MMP $\phi\colon X' \dashrightarrow X^+$ (for $0 < t \leq \varepsilon$), terminating when $\tau = 0$.
Step 3 (properness over U and descent). With $U = X \setminus \operatorname{Null}(\alpha)$ and $X'_U = \nu^{-1}(U)$, every step is vertical over U: $\phi_U = \phi|_{X'_U}$ is a proper bimeromorphic map over U. Moreover the divisors contracted by ϕ coincide with the ν-exceptional divisors, so the induced $\psi\colon X \dashrightarrow X^+$ is a *small* bimeromorphic map of strongly $\mathbb{Q}$-factorial varieties, isomorphic outside $\operatorname{Null}(\alpha)$.
Step 4 (build the contraction). On a common resolution $p\colon W \to X$, $q\colon W \to X^+$ form $E = p^*\eta - q^*\eta^+$; its positivity lets Grauert's theorem contract it, producing $g\colon W \to Z$ and hence induced morphisms $f\colon X \to Z$ and $f^+\colon X^+ \to Z$.
Step 5 (numerical properties). By construction f contracts a nonempty set of α-trivial curves; one checks it contracts *every* curve in R, is $(K_X + B)$-negative, and satisfies $\alpha = f^*\alpha_Z$ (base-point-freeness) with α_Z Kähler.

Step 6 (Kählerness after the flip). Since ψ_* is an isomorphism, $\rho(X^+/Z) = 1$; f^+ contracts a $(K_{X^+} + B^+)$-positive curve, so $K_{X^+} + B^+$ is f^+-ample, and $(f^+)^*\alpha_Z + \delta(K_{X^+} + \Delta^+)$ is Kähler for $0 < \delta \ll 1$; hence X^+ is again a Kähler 3-fold. ■

Remark 6.3 (endpoint)**.** The clean end statement is Theorem 3.2: any $\mathbb{Q}$-factorial gdlt flipping contraction of a Fujiki-class variety admits its flip, locally projective over Z. The generalized-pair and relative refinements for Kähler 3-folds are due to Das–Hacon–Yáñez [DHYn25], who additionally bound the flipping curve by $0 > (K_X + B + \boldsymbol{\beta}_X) \cdot C \geq -6$.

7 The Höring–Peternell approach

The earlier proof, specific to threefolds, contracts the null locus by classical surface-and-bundle geometry rather than a lifted MMP. It handles both contraction types, and in each the argument is a direct analytic contraction with no auxiliary MMP anywhere.

7.1 Divisorial type, contraction to a point ($n(\alpha) = 0$)

The starting data are the same as in Proposition 5.1: a divisorial K_X-negative extremal ray R with $S = \mathrm{Null}(\alpha)$ covered by the curves of R. The divisorial case splits on $n(\alpha) = n(\alpha|_{S^\nu}) \in \{0, 1\}$.

Theorem 7.1 ([HP16, Cor. 7.7])**.** *If R is divisorial and $n(\alpha) = 0$, then S blows down to a point: there is a bimeromorphic morphism $\varphi\colon X \to Y$ to a normal compact threefold with $\dim \varphi(S) = 0$ and $\varphi|_{X\setminus S}$ an isomorphism.*

Proof. Since $n(\alpha) = 0$, $\nu^*\alpha \equiv 0$ on the normalization $\tilde{S}$, so $\alpha \cdot C = 0$ for every curve $C \subset S$, i.e. $\alpha|_S \equiv 0$ and every such $[C] \in R$. Let m be the Cartier index of S. By the surface structure of R [HP16], $-mS$ is strictly positive on R; and α is strictly positive on the rest of $\overline{\mathrm{NA}}(X)$ (the supporting-class property of R). Hence for small $\varepsilon > 0$, $\alpha - \varepsilon[mS]$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$, so by the Kleiman criterion for Kähler 3-folds (Fact 2.2) it is a Kähler class $[\omega]$. As $\alpha|_S \equiv 0$, the Cartier class $[-mS|_S]$ is represented by $\frac{1}{\varepsilon}\omega$, so $\mathcal{O}_S(-mS)$ is ample. Grauert's contractibility theorem (Fact 2.4) then blows S down to a point. ■

7.2 Divisorial type, contraction to a curve ($n(\alpha) = 1$)

Here the nef reduction is a genuine fibration and the surface is contracted along it.

Lemma 7.2 ([HP16, Lem. 7.8])**.** *If R is divisorial and $n(\alpha) = 1$, there is a fibration $f : S \to B$ onto a curve such that a curve $C \subset S$ is contracted iff $[C] \in R$; its general fibre is $\mathbb{P}^1$.*

Proof sketch. Let $\tilde{C}$ be a general fibre of the nef reduction $\tilde{f}\colon \tilde{S} \to \tilde{B}$ of $\nu^*\alpha$; then $\nu^*\alpha \cdot \tilde{C} = 0$, so $\alpha \cdot \nu(\tilde{C}) = 0$ and $C := \nu(\tilde{C})$ has $[C] \in R$, whence $K_X \cdot C < 0$. Because C moves in a positive-dimensional family (divisorial type), the Zariski-decomposition analysis of K_X-negative curves gives $K_S \cdot C < K_X \cdot C < 0$; choosing $\tilde{C}$ through smooth points, $K_S \cdot C \le -2$ and $K_{\tilde{S}} \cdot \tilde{C} \le K_S \cdot C \le -2$. But $\tilde{C}$ is a fibre of $\tilde{f}$ with $\tilde{C}^2 = 0$, so $\deg K_{\tilde{C}} = K_{\tilde{S}} \cdot \tilde{C} \ge -2$; hence equality, $\tilde{C} = C \cong \mathbb{P}^1$ with $K_S \cdot C = -2$. Since $\mathcal{O}_S(C)$ is invertible ($C \subset S_{\mathrm{reg}}$) and $C^2 = 0$ with $K_{\tilde{S}} \cdot C = -2$, asymptotic Riemann–Roch on $\tilde{S}$ (transported to S through the normalization sequence, whose cokernel is supported off C) gives $\kappa(S, \mathcal{O}_S(C)) = 1$; a multiple of C defines the fibration $f\colon S \to B$. ∎

Theorem 7.3 ([HP16, Cor. 7.9], [HP24, Thm. 1.1]). *With $f\colon S \to B$ as in Lemma 7.2, S contracts onto the curve: there is a bimeromorphic morphism $\varphi\colon X \to Y$ to a normal compact threefold with $\varphi|_S = f$ (up to Stein factorization) and $\varphi|_{X\setminus S}$ an isomorphism.*

Idea of proof. Off a finite set $Z \subset B$ the fibration is a $\mathbb{P}^1$-bundle $f^\circ\colon S^\circ \to B^\circ$ with $\mathcal{O}_S(-mS)$ f-ample, so the conormal bundle $N^*_{S^\circ/X^\circ} \cong \mathcal{O}_{S^\circ}(-mS)$ is f°-ample. Nakano's theorem contracts S° fibrewise to B°; a divisorial-extension theorem of Höring–Peternell then globalizes the contraction over the finitely many bad fibres.

Proof sketch. Replacing f by its Stein factorization we may assume $f_*\mathcal{O}_S = \mathcal{O}_B$. As X is $\mathbb{Q}$-factorial, S is $\mathbb{Q}$-Cartier of some index m, and since f contracts an extremal ray, $\mathcal{O}_S(-mS)$ is f-ample. Because X is terminal, $\mathrm{Sing}(X)$ is finite; setting $Z = f(\mathrm{Sing}\, X) \cup \mathrm{Sing}(B)$ and removing $f^{-1}(Z)$, the restriction $f^\circ\colon S^\circ \to B^\circ$ is a $\mathbb{P}^1$-bundle with $N^*_{S^\circ/X^\circ}$ f°-ample. By Nakano's theorem there is $\varphi^\circ\colon X^\circ \to Y^\circ$ realizing f° as the blow-up of Y° along B°. Finally, the three hypotheses of the Höring–Peternell divisorial-contraction theorem [HP24] hold — $\mathcal{O}_S(-mS)$ f-ample, the extension φ° over $X \setminus f^{-1}(Z)$, and the isomorphism $\mathcal{I}^k_{B\setminus Z} \to \varphi^\circ_* \mathcal{I}^k_{S\setminus f^{-1}(Z)}$ at the generic point of B — so the contraction extends to $\varphi\colon X \to Y$. ∎

7.3 Flipping type (terminal 3-folds)

Throughout this subsection X is a *terminal* compact Kähler 3-fold with K_X pseudo-effective, $R = \mathbb{R}^+[\Gamma_{i_0}]$ a small K_X-negative extremal ray, and α its nef supporting class (Fact 2.1). Being terminal, X has only isolated (rational, Gorenstein-in-codimension-2) singularities, so the geometry localizes at the finitely many curves in R.

Idea of proof. The strategy is entirely analytic and avoids any auxiliary MMP. First show the supporting class α is nef and big with 1-dimensional non-Kähler locus C (the curve swept by R); the key numerical input is that $\alpha^2 \cdot S > 0$ for every surface S (Proposition 7.4), so no surface can be α-trivial. Then α is modified Kähler, so a modification splits $\mu^*\alpha = \widetilde{\alpha} + E$; on the preimage of each component of C the class $-E$

restricts to the Kähler class $\widetilde{\alpha}$, giving an *ample conormal bundle*. Grauert's contractibility theorem then contracts C, and the rigidity lemma descends the contraction to X.

Proposition 7.4 (positivity on surfaces; [HP16, Prop. 7.11])**.** *Suppose R is small with nef supporting class α, and let $S \subset X$ be an irreducible surface. Then $\alpha^2 \cdot S > 0$.*

Proof. Since α is nef, $\alpha^2 \cdot S \geq 0$; suppose for contradiction $\alpha^2 \cdot S = 0$. By Fact 2.1 the class $\alpha - K_X$ is positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$ after rescaling, hence Kähler by Fact 2.2; write $\omega := \alpha - K_X$.
If $\alpha|_S = 0$ then $-K_X|_S = \omega|_S$ is ample, so S is projective and covered by curves whose classes lie in R (as $\alpha|_S = 0$), contradicting smallness of R.
So assume $\alpha|_S \neq 0$. From $0 = \alpha^2 \cdot S = K_X \cdot \alpha \cdot S + \omega \cdot \alpha \cdot S$ and $\omega \cdot \alpha \cdot S = \omega|_S \cdot \alpha|_S > 0$ (Hodge index) we get $K_X \cdot \alpha \cdot S < 0$. Let $\pi \colon S' \to S$ be the composition of normalization and minimal resolution, so $K_{S'} + E = \pi^* K_S$ with $E \geq 0$; then $K_{S'} \cdot \pi^* \alpha|_S \leq K_S \cdot \alpha|_S < 0$. Hence $K_{S'}$ is not pseudo-effective, S' is uniruled with $H^2(S', \mathcal{O}_{S'}) = 0$, and $\pi^* \alpha$ is a genuine nef $\mathbb{R}$-divisor on the projective surface S', an element of $\overline{\mathrm{NM}}(S')$ positive on every movable curve (the ray R carries only finitely many curve classes). Fixing an ample A and applying the BDPP decomposition $\pi^* \alpha = C_\varepsilon + \sum \lambda_{i,\varepsilon} M_{i,\varepsilon}$ with $(K_{S'} + \varepsilon A) \cdot C_\varepsilon \geq 0$ and $M_{i,\varepsilon}$ movable: since $(\pi^* \alpha)^2 = 0$ and $\pi^* \alpha \cdot M_{i,\varepsilon} > 0$ we must have $\pi^* \alpha = C_\varepsilon$ for all ε, whence $K_{S'} \cdot \pi^* \alpha \geq 0$ in the limit — contradicting $K_{S'} \cdot \pi^* \alpha < 0$. ∎

Theorem 7.5 (the flipping contraction; [HP16, Thm. 7.12])**.** *Suppose R is small and let C be the locus covered by the curves in R. Then C is contractible: there is a flipping contraction $\varphi \colon X \to Y$ with $\mathrm{Ex}(\varphi) = C$. Consequently, by Theorem 3.2, the flip of φ exists.*

Proof. *α is nef and big.* By Fact 2.1, $-K_X + \alpha$ is positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$ after rescaling, hence Kähler (Fact 2.2). Thus $\alpha = K_X + (-K_X + \alpha)$ is the sum of a pseudo-effective and a Kähler class, so big; combined with nefness, $\alpha^3 > 0$.
The non-Kähler locus is the curve C. Let $\pi \colon \widehat{X} \to X$ be a desingularization. By Fact 2.3, $E_{nK}(\pi^* \alpha)$ is the null locus of the nef and big class $\pi^* \alpha$. If $Z \subset \widehat{X}$ is a surface with $(\pi^* \alpha)^2 \cdot Z = 0$ then, by the projection formula and Proposition 7.4, $\dim \pi(Z) \leq 1$; hence $E_{nK}(\pi^* \alpha)$ is a finite union of π-exceptional surfaces and curves. Since $E_{nK}(\alpha) \subset \pi(E_{nK}(\pi^* \alpha))$ and X_{sing} is finite (terminal $\Rightarrow$ isolated), we conclude $E_{nK}(\alpha) = \bigcup_{\alpha \cdot B = 0} B = C$, a curve.
α is modified Kähler; produce an ample conormal bundle. As the big class α has non-Kähler locus containing no divisor, it is modified Kähler (Fact 2.3): there is $\mu \colon \widetilde{X} \to X$ and a Kähler $\widetilde{\alpha}$ with $\mu^* \alpha = \widetilde{\alpha} + E$, $E \geq 0$ μ-exceptional and $\mu(\mathrm{Supp}\, E) = C$. For a connected component $Z \subset C$ put $D_Z := \mathrm{Supp}\, \mu^{-1}(Z)$. Since $\alpha|_Z \equiv 0$ we get $-E|_{D_Z} = \widetilde{\alpha}|_{D_Z}$, which is ample. Perturbing the coefficients of E (ampleness is open) and clearing denominators, we obtain an integral $E' = \sum m_i D_i + R$ with $D_i \subset D_Z$, R disjoint from D_Z, and $-E'|_{D_Z}$ ample. Hence the conormal bundle $N^*_{\sum m_i D_i / \widehat{X}} = \mathcal{O}(-\sum m_i D_i)$ is ample (ampleness may

be tested on the reduction).
Contract. By Grauert–Fujiki contractibility (Fact 2.4) there is $\psi\colon \widehat{X} \to Y_Z$ contracting $\sum m_i D_i$ to a point; by the rigidity lemma there is $\varphi_Z\colon X \to Y_Z$ with $\psi = \varphi_Z \circ \mu$. Running this over all components of C yields the desired contraction $\varphi\colon X \to Y$ with $\mathrm{Ex}(\varphi) = C$. As φ is small and K_X-negative on R, it is a flipping contraction, and its flip exists by Theorem 3.2. ∎

8 Comparison of the two approaches

Observation 8.1 (Höring–Peternell vs. Das–Hacon)**.** The two methods produce the *same* objects — the divisorial and flipping contractions — by opposite philosophies.

- **Höring–Peternell** is a *direct analytic contraction*. In the divisorial case one contracts the Moishezon surface $S = \mathrm{Null}(\alpha)$ by hand: Kleiman plus Grauert to a point (Theorem 7.1), or a $\mathbb{P}^1$-bundle structure plus Nakano plus a divisorial-extension theorem to a curve (Theorem 7.3). In the flipping case one proves the supporting class is modified Kähler with a 1-dimensional non-Kähler locus (Proposition 7.4), extracts an ample conormal bundle, and contracts by Grauert (Theorem 7.5). It is clean and geometric but uses surface classification, Riemann–Roch, and terminal 3-fold geometry throughout — it is intrinsically 3-dimensional, and the flipping case is tied to the terminal (non-log) setting, where singularities are isolated and the null locus is automatically a curve.
- **Das–Hacon** is an *induction that bootstraps the log MMP*. One passes to a $\mathbb{Q}$-factorial dlt modification, runs an auxiliary MMP steered by nef thresholds on boundary surfaces, feeds each elementary contraction to the all-dimensional PLT contraction theorem Theorem 4.1, and descends. It is heavier machinery but reaches klt/dlt log pairs and generalized pairs, where boundary components and non-isolated centres occur.

Key takeaways

- Both branches localize the whole problem on $\mathrm{Null}(\alpha)$, and the branch is decided by its dimension (Observation 1.4): 2 gives a divisorial contraction, ≤ 1 a flipping one. In the divisorial case the argument then splits again on the nef dimension $n(\alpha|_{S^\nu}) \in \{0, 1\}$ — contraction to a point versus to a curve.
- In the Kähler MMP the *flip* is soft: given a flipping contraction it is Proj_Z of a locally finitely generated relative log canonical algebra (Theorem 3.1, Fact 2.5). The work is always the *contraction*.
- Both methods share the analytic core: realize contractions via *Grauert–Fujiki contractibility + the rigidity lemma*. They diverge only in how the contraction is produced — Höring–Peternell contract *directly* (ample conormal bundle),

Das–Hacon *indirectly* by an auxiliary MMP on a dlt modification.

- Nothing in the Das–Hacon logic is special to dimension three except the inputs it quotes (3-fold termination, the Moishezon-surface structure of Null(α)). Consequently it is the method that survives into higher dimension, while Höring–Peternell remains the more geometric and concrete account in dimension three.
- Reality check: flips are genuinely a *singular / log* 3-fold phenomenon — for smooth projective 3-folds there are no flips; the first honest flips appear on singular or log 3-folds and (e.g. toric) smooth 4-folds.

References

[Bou04] Sébastien Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76.

[DH20] Omprokash Das and Christopher Hacon, *The log minimal model program for kähler 3-folds*, 2020.

[DH24] _____, *On the minimal model program for kähler 3-folds*, 2024.

[DHP24] Omprokash Das, Christopher Hacon, and Mihai Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68. MR 4719824

[DHYn25] Omprokash Das, Christopher Hacon, and José Ignacio Yáñez, *Mmp for generalized pairs on kähler 3-folds*, 2025.

[DP04] Jean-Pierre Demailly and Mihai Păun, *Numerical characterization of the kähler cone of a compact kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274.

[Fuj22] Osamu Fujino, *Minimal model program for projective morphisms between complex analytic spaces*, 2022.

[HP16] Andreas Höring and Thomas Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264. MR 3437871

[HP24] _____, *A contraction theorem for divisors fibering over a curve*, 2024.

Finite Generation in the Kähler MMP

Kähler MMP Yi Li

Contents

1 The geometric meaning of finite generation 1

2 Why a good minimal model supplies the right ring 2

3 The Kähler question 3

4 Research directions 3

Big picture

Finite generation turns infinitely many pluricanonical systems into one projective model. The geometric mechanism is a stable fixed part together with a moving part defining a morphism. We prove this mechanism precisely, then explain why a transcendental class needs a different conclusion.

1 The geometric meaning of finite generation

Let L be a line bundle on a normal projective variety X. Its section ring is $R(X,L) = \bigoplus_{m\geq 0} H^0(X, L^{\otimes m})$, with multiplication of sections. Finite generation says that finitely many sections generate every higher-degree section by sums and products. Passing to a sufficiently divisible Veronese subring removes the different generator degrees.

Theorem 1.1. *Suppose $R(X,L)$ is finitely generated and has a nonzero positive-degree element. There exist an integer $d > 0$ and a projective birational morphism $\mu : Y \to X$, with Y smooth, such that $\mu^*(dL) = P + F$, where P is a basepoint-free Cartier divisor and $F \geq 0$ is Cartier, and multiplication by the canonical section of kF gives an isomorphism*

$$H^0(Y, kP) \simeq H^0(Y, k\mu^*(dL)) \quad \text{for every } k \geq 0.$$

Conversely, the existence of such a decomposition implies finite generation of the Veronese ring $R(X,L)^{(d)}$.

Proof. A standard graded-algebra lemma gives d such that the Veronese ring is generated by its degree-one piece $H^0(X, dL)$. Resolve the base ideal of this linear system. The pulled-back ideal becomes $\mathcal{O}_Y(-F)$, and the residual system defines a basepoint-free divisor $P = \mu^*(dL) - F$.
Because the Veronese ring is generated in degree one, every section of kdL is a sum of products of k sections of dL. Pulling back shows that each is divisible by the canonical section of kF. Thus its quotient is a section of kP. Conversely, multiplying any section of kP by that canonical section produces a section of $k\mu^*(dL)$. These operations are inverse. Here $H^0(Y, k\mu^*(dL)) = H^0(X, kdL)$ follows from $\mu_*\mathcal{O}_Y = \mathcal{O}_X$ and the projection formula. For the converse let $g : Y \to Z \subset \mathbb{P}^N$ be the morphism defined by P, so $\mathcal{O}_Y(P) = g^*\mathcal{O}_Z(1)$. The projection formula identifies $H^0(Y, kP)$ with $H^0(Z, g_*\mathcal{O}_Y \otimes \mathcal{O}_Z(k))$. The graded module of sections of a coherent sheaf twisted by an ample line bundle is finitely generated over the homogeneous coordinate ring after adjoining finitely many low-degree pieces, by Serre's theorem. Apply this to the coherent algebra $g_*\mathcal{O}_Y$. Finite module generators, together with the coordinate-ring generators, generate $R(Y, P)$ as an algebra. The assumed isomorphisms respect multiplication, so they identify this algebra with $R(X, L)^{(d)}$. ∎

The converse is deliberately stated for the Veronese ring: it is exactly what the hypothesis controls. For a semiample line bundle one can prove finite generation of the whole section ring directly, as follows.

Corollary 1.2. *If L is semiample, $R(X, L)$ is finitely generated.*

Proof. Choose d such that $dL = g^*A$ for the morphism $g : X \to Z$ defined by $|dL|$, where $A = \mathcal{O}_Z(1)$ is ample. For $0 \le r < d$, the degree-r residue module is

$$\bigoplus_{k\ge 0} H^0(X, (kd + r)L) = \bigoplus_{k\ge 0} H^0(Z, g_*\mathcal{O}_X(rL) \otimes A^k).$$

Each is finitely generated over the section ring of A, by coherence and Serre's theorem. Their finite direct sum is the whole section ring of L. Hence finitely many module generators and the finitely many generators of the section ring of A generate it as an algebra. ∎

2 Why a good minimal model supplies the right ring

The bridge from the MMP to the preceding result is the following elementary birational calculation. It explains why exceptional discrepancies do not change the canonical ring.

Lemma 2.1. *Let $q : W \to Y$ be a proper birational morphism of normal varieties, A a Cartier divisor on Y, and $E \ge 0$ an integral q-exceptional divisor. Then $q_*\mathcal{O}_W(q^*A + E) = \mathcal{O}_Y(A)$.*

Proof. It suffices to work where A is trivial. A section upstairs is a meromorphic function whose possible poles lie in E. Away from the image of the exceptional locus it is holomorphic. The image of E has codimension at least two in normal Y, so the function extends holomorphically across it. This proves containment in $\mathcal{O}_Y$; the reverse containment follows by pullback of holomorphic functions. ∎

Proposition 2.2. *Suppose a common resolution W of two normal projective pairs has maps $p : W \to X$, $q : W \to Y$ and an equality of rational divisors $p^*(K_X + B) = q^*(K_Y + B_Y) + E$, where $E \geq 0$ is q-exceptional. If $K_Y + B_Y$ is semiample, the sufficiently divisible canonical Veronese ring of (X, B) is finitely generated.*

Proof. Choose d clearing all Cartier indices and denominators. Apply Lemma 2.1 to $kd(K_Y + B_Y)$ and kdE for every $k \geq 0$. The projection formula for p gives compatible identifications $H^0(X, kd(K_X + B)) = H^0(Y, kd(K_Y + B_Y))$. They respect multiplication. The ring on the right is finitely generated by Corollary 1.2. ∎

3 The Kähler question

For a rational adjoint divisor on a compact Kähler space there is still a pluricanonical ring. The problem is to construct a good minimal model or to prove finite generation by analytic means. For an arbitrary real transcendental $(1,1)$-class, there is no associated line bundle and hence no section ring of this form. The geometric substitute is a contraction $f : X \to Z$ and a Kähler class ω_Z with $\alpha = f^*\omega_Z$. This is why transcendental base point freeness is a separate chapter: it constructs the geometric output without a graded algebra of sections as input.

4 Research directions

The essential question is which analytic hypotheses produce the stable moving part used in Theorem 1.1. A divisorial Zariski decomposition only gives modified nefness; it does not give the basepoint-free divisor required in that proof. Another direction is relative finite generation over analytic bases: coherence is local, whereas a single finite list of generators need not work over a noncompact base. Finally, when an adjoint class is transcendental, one should formulate the desired contraction and descent statement directly and identify which part of the projective ring argument it replaces.

References

Termination Problems in the Kähler MMP

Kähler MMP Yi Li

Contents

1 What scaling records 1

2 The positive-limit reduction 1

3 What changes when the limit is zero 2

4 Research directions 3

Big picture

Constructing one MMP step and proving that the process stops are different problems. Scaling records the process by a decreasing sequence of nef thresholds. Its limit distinguishes the region where positivity remains available from the boundary where termination is difficult. We explain this distinction through the change-of-scaling calculation.

1 What scaling records

Write $D = K_X + B$ and choose an effective real Cartier divisor C with $D + C$ nef over the base. In an MMP with scaling, on the ith model put $\lambda_i = \inf\{t \geq 0 : D_i + tC_i \text{ is nef over the base}\}$. At a nonterminal step the chosen extremal ray satisfies

$$D_i \cdot R_i < 0, \qquad (D_i + \lambda_i C_i) \cdot R_i = 0.$$

In particular $\lambda_i > 0$ and $C_i \cdot R_i > 0$. The cone theorem and descent of the supporting class are needed to construct these steps; the definition of an infimum alone supplies no ray. The usual scaling construction gives $1 \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq 0$.

2 The positive-limit reduction

The reason a positive limit is useful is that some of the scaling divisor can then be absorbed permanently into the boundary. The same birational steps remain negative for the new adjoint divisor.

Proposition 2.1. *Let an MMP for D with scaling of C have thresholds λ_i. Fix a with $0 \le a < \inf_i \lambda_i$ and $a < 1$. Set $\widetilde{D} = D + aC$ and $\widetilde{C} = (1-a)C$. The same steps form an MMP for $\widetilde{D}$ with scaling of $\widetilde{C}$, with thresholds $\widetilde{\lambda}_i = (\lambda_i - a)/(1-a)$, provided the relevant pair hypotheses are retained.*

Proof. On each model, $\widetilde{D}_i + t\widetilde{C}_i = D_i + (a + (1-a)t)C_i$. The affine change of parameter is increasing and carries $t \ge 0$ to parameters at least a. Since the old threshold is greater than a, the infimum defining the new threshold is exactly the claimed value. On the chosen ray, the supporting equality gives $D_i \cdot R_i = -\lambda_i C_i \cdot R_i$. Hence

$$\widetilde{D}_i \cdot R_i = -(\lambda_i - a)C_i \cdot R_i < 0.$$

Thus the old step is negative for the new divisor as well and is trivial for its new supporting divisor. For a divisorial contraction the underlying contraction is unchanged. For a flip, relative Picard number one and the supporting relation show that the new adjoint divisor is a positive multiple of the old one over the contraction; the same flipped model therefore has the required relative positivity. ∎

Corollary 2.2. *Suppose termination with scaling is known for the class of pairs under consideration whenever the boundary contains an ample effective divisor. If $C \ge H \ge 0$ with H ample and $\lambda = \lim_i \lambda_i > 0$, then the given MMP terminates, assuming the shifted pair lies in that class.*

Proof. Take $a = \lambda/2$. The new boundary $B + aC$ contains aH, and Proposition 2.1 identifies the original sequence with an MMP for that boundary. Apply the stated termination input. ∎

This is the reduction underlying the positive-limit case of Birkar's termination theorem in the vault. The deep input is termination for a boundary containing an ample part; the parameter change does not prove that input. In particular, applying the corollary in the Kähler setting requires an appropriate analytic termination theorem with the same hypotheses.

3 What changes when the limit is zero

If $\lambda_i \downarrow 0$, no fixed positive multiple of C can be absorbed while preserving negativity of every step. This is the precise point where the preceding argument fails. Nor does convergence of the thresholds imply termination: a decreasing positive real sequence can have infinitely many terms. Even a constant threshold permits flips which are trivial for the supporting divisor.

A proof at the zero limit must therefore add geometric information. Typical inputs are a

minimal model for the limiting pair, special termination near the non-klt locus, or finiteness of models for nearby boundaries. Each controls the birational maps, rather than merely the real numbers λ_i.

Proposition 3.1. *For a relative projective MMP with finite-dimensional $N^1(X/Z)$, there are only finitely many divisorial contractions. If the program is infinite, its tail consists entirely of flips.*

Proof. An elementary divisorial contraction has relative Picard number one; the relative Picard number of the successive model drops by one. A flip is an isomorphism in codimension one between the relative $\mathbb{Q}$-factorial models and preserves their relative Picard number. The Picard number is a nonnegative integer, so it can drop only finitely many times. After the last drop all remaining steps are flips. ∎

For transcendental Kähler programs, the analogous argument requires the appropriate theorem transporting Bott–Chern classes across the steps. One cannot substitute curve numerical spaces without checking that they retain the required information.

4 Research directions

The zero-limit case concentrates the termination problem. A useful question is which extra information about the limiting adjoint class forces a finite set of birational models. Another is how special termination can be proved by adjunction to boundary components in the analytic category. Finally, any proposed scaling argument should identify both the theorem producing its threshold ray and the theorem ruling out an infinite sequence of supporting-class-trivial flips.

References

Transcendental Base Point Freeness

Kähler MMP

Yi Li

Contents

1 What base point freeness should mean 1

2 The adjoint theorem and its two cases 2

3 Why the discrepancy twist is the right sheaf 2

4 Eliminating the exceptional error in the non-big case 3

5 What remains in the central proof 3

6 Research directions 4

Big picture

For a divisor, base point freeness constructs a morphism from sections. A transcendental class has no section ring. The problem is therefore to construct the morphism geometrically and then prove that the class descends to a Kähler class on its target. Vanishing and divisorial Zariski decomposition explain two crucial steps in that passage.

1 What base point freeness should mean

If a line bundle L has a basepoint-free multiple, the associated morphism, after Stein factorization, writes $c_1(L)$ as the pullback of an ample real class. This is the part of the conclusion that still makes sense for a transcendental class. Accordingly, a nef class α is called semiample here when there is a contraction $f : X \to Y$ onto a compact Kähler space and a Kähler class γ such that $\alpha = f^*\gamma$ in Bott–Chern cohomology.

Proposition 1.1. *If $\alpha = f^*\gamma$ with γ Kähler, an irreducible compact curve $C \subset X$ is contracted by f if and only if $\alpha \cdot C = 0$.*

Proof. The projection formula gives $\alpha \cdot C = \gamma \cdot f_* C$. If C is contracted, $f_* C = 0$. Otherwise $f_* C$ is a positive integer multiple of the image curve, on which a Kähler class has positive integral. This proves both implications. ∎

The converse is not a formal criterion for semiampleness. Knowing which curves should be contracted neither constructs the target nor proves descent of a Bott–Chern class. On a Kähler space, positivity on curves alone also does not prove that a class is Kähler.

2 The adjoint theorem and its two cases

The Hacon–Xie theorem, in the form recorded in the vault's paper digest, states: for a compact Kähler generalized klt pair $(X, B + \boldsymbol{\beta})$, if $\alpha = K_X + B + \boldsymbol{\beta}_X$ is nef and $B + \boldsymbol{\beta}_X$ is modified big, then α is semiample. The resulting contraction is Moishezon, and is projective when X is strongly $\mathbb{Q}$-factorial. The precise modified-bigness and strong $\mathbb{Q}$-factoriality hypotheses belong to the theorem; ordinary bigness or ordinary $\mathbb{Q}$-factoriality should not silently replace them.

The proof splits according to whether α is big. In the big case, the null locus is a proper subset, and a separate contraction theorem constructs a birational morphism. One then proves vanishing, descent and positivity on its target. In the non-big case, the argument passes through the MRC fibration and a lower-dimensional base; a relative MMP and the canonical bundle formula produce the setting in which induction applies. The following two results isolate steps that explain why the construction can recover the original class.

3 Why the discrepancy twist is the right sheaf

Suppose $\nu : X' \to X$ is a projective log resolution and $h = f \circ \nu : X' \to Y$ is projective, with X, Y normal. Assume

$$K_{X'} + B' + \beta' = \nu^* \alpha,$$

where B' has SNC support and all coefficients less than one, its nonexceptional coefficients are nonnegative, and β' is Kähler. Assume $\nu^* \alpha$ is numerically trivial over both X and Y. The equalities involving classes below are Bott–Chern equalities, or the relative numerical equalities required by the vanishing input.

Proposition 3.1. *If relative Kawamata–Viehweg vanishing applies on X' over both bases to the fractional-boundary pair $(X', \{B'\})$, then $R^i f_* \mathcal{O}_X = 0$ for every $i > 0$.*

Proof. We need a sheaf whose pushforward to X is $\mathcal{O}_X$ and whose difference from the log canonical divisor is relatively positive. The discrepancy identity suggests $L = -\lfloor B' \rfloor$, since

$$L - (K_{X'} + \{B'\}) = \beta' - \nu^* \alpha.$$

Over either base the right side is numerically equivalent to the Kähler class β', hence has the relative positivity required by the assumed vanishing theorem. It follows that $R^i\nu_*\mathcal{O}_{X'}(L) = R^i h_*\mathcal{O}_{X'}(L) = 0$ for $i > 0$.
All coefficients of B' are less than one, so L is effective. The nonexceptional coefficients of B' lie in $[0, 1)$, so L is ν-exceptional. Normality of X now gives $\nu_*\mathcal{O}_{X'}(L) = \mathcal{O}_X$: a meromorphic function allowed poles only along an exceptional divisor is regular off a codimension-two set downstairs and extends across that set. The Leray spectral sequence for $h = f \circ \nu$ has $E_2^{p,q} = R^p f_*(R^q\nu_*\mathcal{O}_{X'}(L))$. All rows except $q = 0$ vanish, so $R^i f_*\mathcal{O}_X \simeq R^i h_*\mathcal{O}_{X'}(L) = 0$. ∎

The twist performs two jobs simultaneously: it absorbs the integral discrepancy into the sheaf, and disappears under pushforward. The theorem producing the birational contraction and the theorem descending the class to its target are additional inputs; neither is proved by this spectral-sequence argument.

4 Eliminating the exceptional error in the non-big case

The non-big construction can end with semiampleness of a class of the form $c\alpha_m + F_m$, rather than of α_m itself. Subtracting an effective divisor does not preserve nefness. The way to remove F_m is to identify it as a negative part throughout the birational process.

Lemma 4.1. *Let $\mu : W \to X$ be a modification, α nef, $c > 0$, and $F \geq 0$ exceptional. Assume the exceptional pullback formula gives $N(c\mu^*\alpha + F) = F$. Suppose a sequence of birational steps transports this class to $\delta_m = c\alpha_m + F_m$, where F_m is the effective pushforward of F, and suppose the negative-part transformation theorem gives $F_m \leq N(\delta_m)$. If δ_m is nef, then $F_m = 0$. If δ_m is semiample, then α_m is semiample as well.*

Proof. A nef class has zero minimal multiplicity along every divisor: its smooth representatives with arbitrarily small negative lower bound have zero Lelong numbers. Hence $N(\delta_m) = 0$. The inequalities $0 \leq F_m \leq N(\delta_m)$ force $F_m = 0$. If $\delta_m = g^*\omega$ with ω Kähler, the remaining identity $c\alpha_m = \delta_m$ gives $\alpha_m = g^*(\omega/c)$, proving semiampleness. ∎

The lemma uses the exceptional pullback and transformation formulas as explicit hypotheses. Its application requires checking them for the particular MMP. It also concerns the final pushforward F_m; it does not say that the original exceptional divisor F was zero.

5 What remains in the central proof

The big case still requires the construction of the contraction by induction on multiplier-ideal thickenings, together with the descent and Kählerness criterion on the target. The non-big case requires the relative MMP over the MRC base, the canonical bundle formula, lifting the base MMP, and transport of semiampleness back across the resulting α-trivial maps.

These are substantial parts of the theorem. The present draft proves the two mechanisms above, and does not present them as a complete proof of Hacon–Xie's theorem. The source digest also records an outstanding check of the dimension-induction dependencies.

6 Research directions

One direction is to isolate the weakest descent theorem needed after constructing a Moishezon contraction. Another is to understand how much of the non-big argument survives when modified bigness is weakened: that hypothesis supports both the relative MMP and the lower-dimensional adjoint structure. A third is to compare the geometric transport of a semiample class across birational maps with the projective argument using invariant section rings. The point is to identify precisely which descent and factorization results replace those rings.

References

Existence of Minimal Models

Kähler MMP

Yi Li

Contents

1 What the output of the program means 1

2 Threefolds: assembling the existence theorem 2

3 Effective fourfolds: why a section changes the problem 3

4 From existence to a good model 3

5 Research directions 3

Big picture

The existence problem asks whether repeated improvement of an adjoint divisor reaches a nef model. It has three separate ingredients: existence of each step, preservation of the category, and termination. Abundance then asks for a morphism defined by the nef divisor. We organize the known threefold and effective fourfold results around this distinction.

1 What the output of the program means

Let (X, B) be a normal compact Kähler pair with $K_X + B$ rational Cartier. A log minimal model has nef adjoint divisor and improves discrepancies for the divisors it contracts. A good minimal model has semiample adjoint divisor. Thus "good" asks for sections, not just nonnegative intersections. If the adjoint divisor is big, a good model also produces its log canonical model by contracting the curves on which that divisor vanishes.

Here is the comparison that makes these models useful. Suppose $\phi : X \dashrightarrow Y$ is a log minimal model and $p : V \to X$, $q : V \to Y$ resolve its graph. The defining negativity condition gives

$$p^*(K_X + B) = q^*(K_Y + B_Y) + E, \qquad E \geq 0, \quad E \text{ is } q\text{-exceptional}.$$

For sufficiently divisible m, this identifies the spaces of m-pluricanonical sections on the two models. Indeed, a rational function which is a section of $\mathcal{O}_V(q^*m(K_Y + B_Y) + mE)$ has no

pole along any divisor downstairs beyond that allowed by $m(K_Y + B_Y)$. Normality extends it across the codimension-two image of E. Conversely a section downstairs pulls back and can be multiplied by the canonical section of mE. These inverse operations respect multiplication. This is why replacing X by a minimal model does not change the relevant Veronese canonical ring.

2 Threefolds: assembling the existence theorem

Theorem 2.1 (Höring–Peternell; Das–Hacon)**.** *Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler dlt threefold pair. If $K_X + B$ is pseudoeffective, a finite sequence of flips and divisorial contractions ends with nef adjoint divisor. For klt pairs with non-pseudoeffective adjoint divisor, an MMP can be chosen ending in a Mori fibre space. See [HP16, DH25, DH24].*

The central point is how the earlier chapters fit together. The cone theorem detects a negative ray when nefness fails; contraction and flip existence construct the next space; preservation results keep it Kähler. Termination is a further theorem, not a consequence of the cone theorem. Divisorial contractions lower the Picard number, but flips do not.

Proposition 2.2. *Suppose in a class of pairs that negative rays can be contracted, flips exist, the class is preserved, and the chosen MMP terminates. Then its output is a nef model or a Mori fibre space. If the adjoint class is pseudoeffective throughout the program, the latter is impossible.*

Proof. At a terminal stage a non-nef adjoint divisor would give a negative ray and another step, unless the contraction has fibre type. Thus the only possible outputs are those stated. For the last assertion, resolve the total space of a fibre-type contraction and pull back a positive current in its adjoint class. Restrict to a sufficiently general smooth fibre of the composite morphism, where the local potentials are not identically minus infinity. This fibre resolves the corresponding projective fibre downstairs. The pulled-back adjoint divisor is the negative of an ample class pulled back from that fibre, up to a positive scale. Its intersection with the appropriate power of the same pulled-back polarization is strictly negative. On the other hand, that power is represented by a smooth semipositive form, so its intersection with the restricted positive current is nonnegative. This is a contradiction. Birational pushforward preserves pseudoeffectivity, so this applies at the last stage of an MMP started with pseudoeffective adjoint class. ∎

This proves the assembly of Theorem 2.1 from the threefold step and termination theorems. The difficulty argument for termination belongs to the termination chapter; an arbitrary count of exceptional divisors on one resolution is not a replacement for it.

3 Effective fourfolds: why a section changes the problem

> **Theorem 3.1** (Das–Hacon–Păun, Theorem 1.1)**.** *Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler dlt fourfold pair with rational boundary. If $K_X + B \sim_{\mathbb{Q}} M \geq 0$, then (X, B) has a log minimal model [DHP24].*

An effective representative supplies a geometric locus on which negative curves must lie. For an effective rational Cartier divisor M and a curve C not contained in its support, a positive Cartier multiple restricts to a line bundle with a nonzero section on the normalization of C. Its degree is nonnegative. Consequently $M \cdot C < 0$ forces $C \subset \operatorname{Supp} M$.

The proof exploits this support, boundary modifications, and lower dimensional adjunction to control the program; the effective-pair termination and construction results in [DHP24, Sections 6–7] are the deep input. This is why the hypothesis is linear effectivity, not merely pseudoeffectivity. A positive current does not supply a finite divisor support on which to run the same induction.

4 From existence to a good model

For a threefold rational pair, applying log abundance to the nef output makes the minimal model good. This is a two-theorem argument: existence first, abundance second. In higher dimension neither termination nor abundance follows from analytic BCHM for projective morphisms, because the required global Kähler contractions and nonvanishing still have to be supplied.

The maximal Albanese setting offers a different source of geometry: the Albanese morphism can organize the relative problem, but its projectivity and the hypotheses of the applicable existence theorem must be checked. It does not remove the distinction between a projective morphism and a projective total space.

5 Research directions

The effective fourfold theorem suggests the precise nonvanishing question needed to extend its scope: when does a pseudoeffective adjoint class admit an effective representative in its linear class? A second question concerns termination when scaling thresholds tend to zero. A third is how relative good models over an Albanese or MRC base can be made into a global good model. These are separate bridges; progress on one should not be recorded as a solution of all three.

References

[DH24] O. Das and C.D. Hacon, *Transcendental minimal model program for projective varieties*, 2024.

[DH25] Omprokash Das and Christopher Hacon, *The log minimal model program for Kähler 3-folds*, J. Differential Geom. **130** (2025), no. 1, 151–207.

[DHP24] O. Das, C.D. Hacon, and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68.

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.

Canonical Bundle Formulas and Subadjunction

Kähler MMP Yi Li

Contents

1 Why a boundary alone is not enough 1

2 The moduli part lives on a higher model 2

3 Subadjunction: restrict first, then descend 3

4 The Kähler interface and research directions 3

Big picture

Adjunction lowers dimension along a divisor. The canonical bundle formula lowers dimension along a fibration. Its two terms record different phenomena: the discriminant measures singular fibres, and the moduli part records variation that remains after those singularities have been accounted for. Subadjunction combines these constructions.

1 Why a boundary alone is not enough

For a contraction $f : (X, B) \to Z$ with $K_X + B \sim_{\mathbb{Q}} f^*L$, we seek a formula on the base. Work first with projective klt-trivial fibrations: the pair is sub-klt over the generic point of Z, and the usual rank-one condition $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) = 1$ is imposed, with the discrepancy b-divisor interpreted on a resolution. It ensures that the construction describes a single adjoint system on the generic fibre. Merely having Kodaira dimension zero on that fibre requires a preliminary reduction to this setting.

For each prime divisor P on the base put

$$t_P = \sup\{t \mid (X, B + t f^*P) \text{ is sub-lc over } \eta_P\}, \quad B_Z = \sum_P (1 - t_P)P, \quad M_Z = L - K_Z - B_Z.$$

The formula $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$ is now a definition of the residual class. Its positivity is a theorem.

Proposition 1.1. *Suppose that over the generic point of P the total space is smooth, $f^*P = \sum_i m_i D_i$, and this divisor together with B has simple normal crossings. Write $b_i = \operatorname{coeff}_{D_i} B$. Then $t_P = \min_i (1 - b_i)/m_i$.*

Proof. In a simple normal crossing subpair, sub-log-canonicity is equivalent to every boundary coefficient being at most one. Adding $t f^*P$ changes the coefficient of D_i to $b_i + t m_i$. The simultaneous inequalities $b_i + t m_i \le 1$ hold precisely for $t \le \min_i (1 - b_i)/m_i$. No additional valuation gives a stricter condition for an SNC subpair. This proves the formula. ∎

For a reduced smooth fibre and zero boundary this gives coefficient zero. For a fibre mD with D smooth generically it gives $1 - 1/m$. Thus the definition is forced by the lc threshold, rather than chosen to resemble the familiar elliptic-surface formula.

2 The moduli part lives on a higher model

Theorem 2.1 (Ambro's moduli theorem, projective klt-trivial setting)**.** *For a projective klt-trivial fibration in the above setting, the moduli b-divisor is b-nef and rational b-Cartier: on a sufficiently high projective birational model $Z' \to Z$, its trace $M_{Z'}$ is nef and its traces on higher models are pullbacks. See [Amb05].*

The proof uses the Hodge-theoretic positivity of the direct image associated with the generically trivial adjoint bundle, after suitable coverings and modification. The threshold calculation separates the boundary singularities from that positivity. This Hodge-theoretic theorem is a supporting input in this base draft.

The higher model matters: M_Z need not itself be nef. On another base model one recomputes the discriminant from the pulled-back fibration; its exceptional coefficients cannot be recovered just by taking strict transforms of B_Z. This is precisely the reason for generalized-pair notation $K_Z + B_Z + \mathbf{M}_Z$.

Proposition 2.2 (The section-ring mechanism)**.** *Suppose $f : X \to Z$ is a contraction of normal spaces, $D \sim_{\mathbb{Q}} f^*L + E$, with D, L, E rational Cartier and $E \ge 0$, and the canonical inclusion $\mathcal{O}_Z \to f_*\mathcal{O}_X(mE)$ is an isomorphism for all sufficiently divisible m. Then the corresponding Veronese section rings of D and L are naturally isomorphic.*

Proof. Choose one integer clearing all denominators and the rational linear equivalence. For every multiple m of it, the projection formula gives $f_*\mathcal{O}_X(mD) = \mathcal{O}_Z(mL) \otimes f_*\mathcal{O}_X(mE) = \mathcal{O}_Z(mL)$. The identifications are induced by the same fixed rational equivalence and canonical sections of mE, so commute with products. Taking global sections proves the assertion. ∎

The direct-image condition is the substantive hypothesis: an arbitrary effective vertical error cannot simply be discarded. Fujino's analytic canonical bundle formula arranges such control after modifications in the relative Iitaka setting; see [Fuj22, Section 21].

3 Subadjunction: restrict first, then descend

For a smooth divisor S on a smooth variety, the residue sequence gives $(K_X + S)|_S = K_S$. If $S+B$ has SNC support and S is not a component of B, restriction gives $(K_X+S+B)|_S = K_S+B|_S$. The singular dlt version replaces $B|_S$ by the different on the normalization of S.

A higher-codimensional lc centre cannot be handled by this elementary restriction alone. On a dlt modification choose a stratum dominating the centre, apply divisorial adjunction along the stratum, and then apply a canonical bundle formula to its morphism to the centre. Stein factorization and any finite descent must be included in this construction. The resulting moduli part explains why subadjunction naturally produces a generalized pair.

> **Theorem 3.1** (Projective subadjunction)**.** *Let (X, B) be a projective lc rational pair, W a minimal lc centre, and A an ample rational divisor. For any rational $\varepsilon > 0$ there is an effective rational divisor B_W such that (W, B_W) is klt and $(K_X + B + \varepsilon A)|_W \sim_{\mathbb{Q}} K_W + B_W$. Here normality of W is part of the subadjunction theorem; see [HK10].*

The small ample term allows the positive contribution to be represented by a general effective boundary while preserving klt singularities. It should not be deleted when quoting this ordinary-pair formulation.

4 The Kähler interface and research directions

The relative projective analytic formula and generalized Kähler formulas in [HP24] are the interfaces used in the later MMP. Their hypotheses include the nature of the morphism and the positivity of the nef data. The projective theorem above does not by itself give an unrestricted formula for every proper Kähler fibration.

The key questions concern positivity and descent of the moduli class, and whether it is semiample on an appropriate model. Nefness and semiampleness are distinct claims. For real transcendental classes, positivity only on curves does not characterize Kähler nefness; an extension of the projective proof must control the actual class, not merely its intersections with algebraic curves.

References

[Amb05] F. Ambro, *The moduli b-divisor of an lc-trivial fibration*, Compos. Math. **141** (2005), no. 2, 385–403.

[Fuj22] O. Fujino, *Minimal model program for projective morphisms between complex analytic spaces*.

[HK10] C.D. Hacon and S.J. Kovács, *Classification of higher dimensional algebraic varieties*, Oberwolfach Seminars, vol. 41, Birkhäuser Verlag, Basel, 2010.

[HP24] C.D. Hacon and M. Păun, *On the canonical bundle formula and adjunction for generalized kaehler pairs*, 2024.

Uniruled Kähler Manifolds and the MRC Fibration

Kähler MMP

Yi Li

Contents

1 The canonical class detects covering rational curves 1

2 The MRC quotient and the projective bridge 2

3 A concrete model: a ruled space over a non-uniruled base 2

4 Role in the MMP and research directions 3

Big picture

Uniruledness detects the negative side of the canonical class. The MRC fibration separates the rationally connected directions from the non-uniruled base. Its importance for the Kähler MMP is practical: after choosing a suitable model, this fibration is projective, so the relative tools developed for analytic projective morphisms become available even when the total space is not projective.

1 The canonical class detects covering rational curves

Theorem 1.1 (BDPP; Ou in the compact Kähler case)**.** *A smooth compact Kähler manifold X is uniruled if and only if K_X is not pseudoeffective. See [BDPP13, Ou25] and the rational curves chapter for the proof architecture.*

The nontrivial direction produces covering rational curves from failure of positivity. It is not enough to find one curve of negative canonical degree: exceptional rational curves on a non-uniruled variety already show the distinction.

One must also distinguish K_X from an adjoint divisor. On $\mathbb{P}^1$, a boundary consisting of four distinct points each with coefficient $1/2$ is klt and makes $K_{\mathbb{P}^1} + B$ have degree zero. Thus an uniruled space can carry a pseudoeffective, even trivial, adjoint divisor. The MMP for pairs follows $K_X + B$.

2 The MRC quotient and the projective bridge

The maximal rationally connected quotient $X \dashrightarrow Z$ has rationally connected general fibres, and a rational curve through a very general point of X stays in its fibre. This maximality is the property used below. Resolve the graph and the base to obtain a smooth holomorphic Kähler model $f : X' \to Z'$.

Theorem 2.1 (Claudon–Höring)**.** *Every smooth holomorphic model of the MRC fibration of a compact Kähler manifold is a projective morphism. More generally a compact Kähler space with klt singularities has a projective model of its MRC fibration [CH24].*

The underlying projectivity criterion says that a fibration between compact Kähler manifolds is projective when pullback induces an isomorphism on holomorphic two-forms. For the MRC fibration the rationally connected fibres have no such forms; the descent theorem for forms gives the required isomorphism. Both descent and the projectivity criterion are geometric inputs, not consequences of projectivity of each individual fibre.

Proposition 2.2. *The smooth base Z' of such a model is not uniruled. Consequently $K_{Z'}$ is pseudoeffective.*

Proof. Suppose instead that rational curves cover Z'. Choose one through a very general point and meeting the locus of rationally connected fibres. Normalize it, and form the main component of the pullback of X' which dominates this curve. Resolve this component. Since f is projective, the resulting smooth total space is projective over $\mathbb{P}^1$, hence projective. Its general fibre is rationally connected. The theorem of Graber–Harris–Starr, together with the usual rational-connectedness consequence, makes this total space rationally connected: its base and general fibre are rationally connected [GHS03].
Two very general points lying over different points of this rational curve can therefore be joined by a rational curve (or a rational chain suffices). Its image in X' connects different general MRC fibres, contradicting maximality. Thus Z' is not uniruled, and Theorem 1.1 gives the last assertion. ∎

3 A concrete model: a ruled space over a non-uniruled base

Let Z be a smooth compact Kähler manifold containing no rational curves, for example a complex torus, and put $X = \mathbb{P}^1 \times Z$. Any map $\mathbb{P}^1 \to X$ is constant after projection to Z. Each fibre of $X \to Z$ is rationally connected, so this projection is exactly the MRC quotient. It is projective even when Z, and hence X, is nonprojective.

For $F = \mathbb{P}^1 \times \{z\}$ one has $K_X \cdot F = -2$. If ω is a Kähler class, the equation $(K_X + t\omega) \cdot F = 0$ forces $t = 2/(\omega \cdot F)$. This illustrates how an adjoint class can become trivial in the rationally

connected direction. It does not prove that the class is globally nef or descends: those require the contraction and descent theorems in the base-point-free chapter.

4 Role in the MMP and research directions

The MRC base and a Mori-fibre-space base need not coincide. The former records all rational connectivity, while the latter is an elementary output of a chosen adjoint MMP and may itself be uniruled. The MRC construction is useful before running a relative program; it is not a substitute for that program.

For the transcendental base-point-free problem, the important questions are whether the adjoint class is trivial on the appropriate fibres, whether it descends as a Bott–Chern class, and whether the descended class is positive enough for induction. Projectivity of the MRC model addresses the available relative tools. It does not settle those three assertions by itself.

References

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[CH24] Benoît Claudon and Andreas Höring, *Projectivity criteria for kähler morphisms*, 2024.

[GHS03] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67.

[Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

Analytic BCHM for Projective Morphisms

Kähler MMP

Yi Li

Contents

1 The theorem and its analytic neighborhood 1

2 The proof in four steps 2

3 The two consequences used later 2

4 How this chapter is used in the series 3

Big picture

Analytic BCHM supplies the projective relative machinery used throughout the Kähler MMP. Projectivity of the morphism is retained; the base is allowed to be analytic. This chapter records the theorem, the logical structure of Fujino's proof, and the passage from good models to finite generation and flips. The longer coefficient and polytope arguments are kept in the separate detailed companion.

1 The theorem and its analytic neighborhood

Let $\pi : X \to Y$ be projective, and let $W \subset Y$ be compact. Fujino's condition (P) asks that X be normal, Y Stein, W a Stein compact, and $W \cap Z$ have finitely many connected components for every analytic subset Z defined near W. A Stein compact has a fundamental system of Stein neighborhoods. These hypotheses supply the noetherian control needed while shrinking around W.

Theorem 1.1 (Fujino, Theorem 1.8)**.** *In this setting, a klt pair (X, Δ) has a log terminal model over a neighborhood of W if Δ is relatively big and $K_X + \Delta$ is relatively pseudoeffective, or if $K_X + \Delta$ is relatively big. In the latter case a log canonical model exists. For rational adjoint divisors the relative canonical algebra is locally finitely generated; see [Fuj22].*

The finite-generation conclusion is local on the base. It does not assert finite generation of global sections over every noncompact Stein open set. Nor does the theorem construct arbitrary global Kähler contractions: the morphism to which it applies is already projective.

2 The proof in four steps

First, nonvanishing is reduced to a sufficiently general projective fibre, where algebraic BCHM applies. Analytic cohomology and base change show that a relevant direct image is nonzero; Cartan's theorem A supplies sections over the Stein base. For real coefficients one does this simultaneously for countably many rational combinations, then uses rational polyhedral approximation. This is Fujino's promotion lemma, not extension of an arbitrary special-fibre section.

Second, the induction constructs good models from effective adjoint divisors. It raises boundary coefficients until a coefficient reaches one. Adjunction to that component makes lower-dimensional finiteness available; pl-flips and special finiteness control the resulting program. The intermediate "neutral model" must then be shown to contract precisely the appropriate fixed divisors. This discrepancy comparison is necessary to obtain a genuine minimal model.

Third, finiteness of models is an induction on the dimension of a boundary polytope. At a parameter Δ_0 with $K_X + \Delta_0 \sim_{\mathbb{R},T} 0$, a point on the segment to Δ satisfies

$$K_X + ((1-t)\Delta_0 + t\Delta) \sim_{\mathbb{R},T} t(K_X + \Delta), \qquad t > 0.$$

Positive scaling preserves nefness and the discrepancy inequalities defining the model, so one reduces to proper faces. In general one first takes the semiample contraction of a central good model; over its target the central adjoint class becomes trivial. Local finiteness of the relevant nef inequalities then brings the models back to the original base. Compactness is used only after this local construction.

Finally, good models give finite generation, and canonical models give flips. In Fujino's notation the induction is

$$F_{n-1} \Rightarrow A_n, \quad E_{n-1} \Rightarrow B_n, \quad (A_n, B_n) \Rightarrow G_n \Rightarrow (E_n, F_n).$$

Here A denotes pl-flips, B special finiteness, G good models, E finiteness of models, and F finite generation. Nonvanishing is proved independently. The strengthened theorem G permits an intermediate projective base while all shrinkings occur in the original Stein base. That strengthening is what lets the induction close.

3 The two consequences used later

> **Proposition 3.1.** *A good minimal model identifies the canonical algebras of the original pair and its model after taking a sufficiently divisible Veronese. That Veronese algebra is locally finitely generated.*

Proof. On a common resolution the pulled-back adjoint divisor of the original pair equals that of the model plus an effective divisor exceptional over the model. Normality gives $q_*\mathcal{O}(mE) = \mathcal{O}$ for divisible m: sections cannot acquire a pole along a divisor downstairs from an exceptional divisor upstairs. Projection formula therefore identifies the two Veronese algebras, including their multiplication.
On the good model a positive multiple of the adjoint divisor is the pullback of a relatively ample line bundle under its semiample contraction. The relative section algebra of that ample bundle is locally finitely generated, and connected fibres identify it with the Veronese algebra in question. ∎

Passing from the Veronese to the full canonical algebra uses Fujino's truncation lemma. This algebraic step is a cited input here, rather than an assertion that every graded algebra is finite over its Veronese.

For a projective flipping contraction, the flip is the relative log canonical model. Construct it on sufficiently small Stein opens in the target. Uniqueness of the canonical model identifies the local constructions on overlaps; uniqueness also gives the cocycle identity. They therefore glue to the flip. This is the local-to-global content of Fujino's flip-existence argument.

4 How this chapter is used in the series

Once a Kähler argument has produced a projective morphism, this theorem gives the relative birational tools. Producing that morphism, preserving Kählerness, and proving transcendental descent remain the tasks of the contraction, projectivity, and base-point-free chapters. The alternative finite-generation approach of Das–Hacon–Păun uses restriction to coefficient-one components and division by their canonical sections; see [DHP24].

The main research interfaces are uniform control of neighborhoods, the availability of projective relative models, and extension or positivity statements with weaker ample hypotheses. The detailed companion retains expanded arguments and their verification notes; they are not prerequisites for reading the rest of this base series.

References

[DHP24] O. Das, C.D. Hacon, and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68.

[Fuj22] Osamu Fujino, *Minimal model program for projective morphisms between complex analytic spaces*, 2022.

Abundance for Kähler Threefolds

Kähler MMP

Yi Li

Contents

1 The theorem and what it adds to the MMP 1

2 Why nonvanishing is a separate theorem 2

3 The proof architecture in dimension three 2

4 Consequences and research directions 3

Big picture

A minimal model has a nef adjoint divisor. Abundance says that this numerical positivity comes from a morphism. The hard part is producing enough sections when the divisor is not big. In dimension three the numerical-dimension-two case exposes the central mechanism: Chern-class inequalities, Euler characteristics, and adjunction together force growth of pluricanonical sections.

1 The theorem and what it adds to the MMP

Throughout this note boundaries are rational and adjoint divisors are rational Cartier; semiampleness means generation of a positive integral multiple. This avoids mixing the line-bundle statement with a transcendental notion of semiampleness.

Theorem 1.1 (Abundance for compact Kähler threefolds)**.** *Let (X, B) be a log canonical compact Kähler threefold pair. If $K_X + B$ is nef, then it is semiample. The terminal boundary-free case was developed in [CHP16]; log abundance follows from [DO24] and the numerical-dimension-two theorem of [DO26].*

The 2026 paper supplies both the orbifold Chern-class inequality and the comparison with ordinary Chern classes needed in the argument. Thus the numerical-dimension-two case should not be summarized as following from an inequality alone.

Proposition 1.2. *Let L be a semiample line bundle on a smooth compact Kähler n-fold. Then $\kappa(L) = \nu(L) = \dim Z$, where $f : X \to Z$ is the contraction defined by a sufficiently divisible power of L.*

Proof. Write $L^m = f^*A$ with A ample on the projective normal space Z and $f_*\mathcal{O}_X = \mathcal{O}_Z$. If $d = \dim Z$, the projection formula identifies $H^0(X, L^{mk})$ with $H^0(Z, A^k)$, whose growth has degree d. Thus $\kappa(L) = d$. For a Kähler form ω on X, the class $f^*c_1(A)^d \cdot [\omega]^{n-d}$ is positive: on the locus where f has rank d, its representing semipositive form has strictly positive wedge with ω^{n-d}. Higher powers of $f^*c_1(A)$ vanish because the base has dimension d. This is exactly $\nu(L) = d$. The case $d = 0$ gives both dimensions zero directly. ∎

This is the numerical meaning of abundance: the dimension predicted by intersections agrees with the dimension visible to sections.

2 Why nonvanishing is a separate theorem

A non-torsion degree-zero line bundle P on an elliptic curve is nef but has no section in any positive power. Indeed, a nonzero section of P^m would have an effective zero divisor of degree zero, hence would be nowhere vanishing and trivialize P^m. This contradicts non-torsion. The adjoint hypothesis in abundance is therefore essential; nefness of an arbitrary line bundle does not suffice.

For nef rational Cartier divisors on smooth compact Kähler spaces, nonvanishing completely handles numerical dimension zero.

Lemma 2.1. *If D is nef with $\nu(D) = 0$ and $mD \sim E \geq 0$ for some positive integer m, then $mD \sim 0$.*

Proof. For a Kähler class ω, numerical dimension zero gives $D \cdot \omega^{n-1} = 0$. If E were nonzero, integration of the strictly positive volume form ω^{n-1} over its prime components would give $E \cdot \omega^{n-1} > 0$. But this number is $mD \cdot \omega^{n-1} = 0$. Hence $E = 0$, as required. ∎

For singular adjoint pairs one uses the corresponding intersection theory and reduction results; the lemma isolates the elementary final step, not the nonvanishing theorem itself.

3 The proof architecture in dimension three

Put $D = K_X + B$. The nonvanishing theorem gives $\kappa(D) \geq 0$. Once $\kappa(D) > 0$, the Iitaka fibration lowers the dimension and the threefold abundance arguments use abundance on curves and surfaces, together with relative MMP and extension or descent of sections.

These reductions are proved in [DO24]. They are stronger than the formal existence of an Iitaka map.

The remaining task is to exclude $\kappa(D) = 0$ when $\nu(D) > 0$. The cases $\nu = 0, 1, 3$ are handled in the earlier work; the case $\nu = 2$ is [DO26, Theorem 1.2]. Its proof separates two situations. If lowering the boundary slightly retains pseudoeffectivity, one prepares a model suitable for Chern-class and cohomological estimates. After disposing of the projective case by projective log abundance, if every such decrease destroys pseudoeffectivity, an adjoint-trivial MMP on the remaining nonprojective threefold produces a Mori fibration over a surface; adjunction and the relative geometry then give a lower-dimensional route. These reductions preserve the Kodaira and numerical dimensions, as checked in Section 7 of that paper.

Here is the cohomological mechanism in the first route, separated from the technical construction of the model. The notation S denotes the effective boundary used after that preparation, not an arbitrary divisor on the original threefold.

Lemma 3.1. *Let L be a line bundle on a compact three-dimensional complex space, and S an effective Cartier divisor. Suppose, for all large n, that $\chi(X, L^n)$ is bounded below, $h^2(X, L^n)$ and $h^2(X, L^n(-S))$ are bounded above, and $h^1(S, L^n|_S) \geq an - b$ with $a > 0$. Then $h^0(X, L^n)$ grows at least linearly, so $\kappa(L) \geq 1$.*

Proof. The restriction sequence gives $H^1(X, L^n) \to H^1(S, L^n|_S) \to H^2(X, L^n(-S))$. Consequently $h^1(X, L^n) \geq an - b - C$ for a constant C. The Euler characteristic identity gives

$$h^0(X, L^n) = \chi(X, L^n) + h^1(X, L^n) - h^2(X, L^n) + h^3(X, L^n).$$

The last term is nonnegative, so the assumed bounds imply $h^0(X, L^n) \geq an - C'$. ∎

In the prepared adjoint setting the same argument uses the coherent ideal sheaf of the boundary when it is only a Weil divisor. The source proves the necessary boundedness statements. Chern-class inequalities control the Euler characteristic; if it already grows, sections grow directly. Otherwise adjunction to the boundary and surface abundance force growth of its first cohomology, and the displayed exact sequence transfers that growth back to X. This is the heart of [DO26, Section 8], with its model construction and cohomological estimates retained as supporting inputs in this draft.

4 Consequences and research directions

Combining threefold minimal-model existence with Theorem 1.1 gives a good model for pseudoeffective rational adjoint threefold pairs in the scope of those existence theorems. The semiample morphism and the exceptional-section comparison then give finite generation of the corresponding canonical ring.

For higher dimensions the instructive questions are which Chern-class estimates survive, how to compare singular and resolved invariants, and how to obtain section growth without projective hyperplane sections. For transcendental classes there is an additional issue: there is no pluricanonical section ring for an arbitrary real class. A proposed analogue must first specify the geometric meaning of its conclusion.

References

[CHP16] F. Campana, A. Höring, and T. Peternell, *Abundance for Kähler threefolds*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 971–1025.

[DO24] O. Das and W.-H. Ou, *On the log abundance for compact Kähler threefolds*, Manuscripta Math. **173** (2024), no. 1-2, 341–404.

[DO26] Omprokash Das and Wenhao Ou, *On the log abundance for compact Kähler threefolds ii*, Proceedings of the London Mathematical Society **132** (2026), e70141.

Tools from Kähler Geometry to Projective Morphisms

Kähler MMP Yi Li

Contents

1 From holomorphic forms to an ample line bundle 1

2 The relative criterion 2

3 Stein localization and descent of sections 2

4 How to use the toolkit 3

Big picture

Many Kähler arguments become manageable after finding a projective morphism. The useful reduction is often relative: the base can remain nonprojective. This toolkit records three mechanisms used repeatedly in the series and explains exactly what each one supplies.

1 From holomorphic forms to an ample line bundle

Theorem 1.1 (Kodaira's criterion)**.** *A smooth compact Kähler manifold X with $H^0(X, \Omega_X^2) = 0$ is projective.*

Proof. Hodge symmetry gives $H^{0,2}(X) = 0$ as well, so $H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})$. The Kähler cone is open in this finite-dimensional space. Rational cohomology is dense, so a Kähler class can be approximated by a rational Kähler class. A positive integral multiple is an integral class of type $(1,1)$, hence is $c_1(L)$ for a holomorphic line bundle by the Lefschetz $(1,1)$ theorem. The $\partial\bar{\partial}$ lemma gives a Hermitian metric on L whose curvature is the chosen positive form. Kodaira embedding now makes X projective. ∎

The vanishing hypothesis makes rational approximation possible in the whole second cohomology. On a general Kähler manifold, rational classes of type $(1,1)$ need not be dense in the Kähler cone.

2 The relative criterion

Theorem 2.1 (Claudon–Höring, Theorem 1.1)**.** *Let $f : X \to Y$ be a fibration between compact Kähler manifolds. If $f^* : H^0(Y, \Omega_Y^2) \to H^0(X, \Omega_X^2)$ is an isomorphism, or if X has an f-big line bundle, then f is projective [CH24].*

The two hypotheses have different proofs. The first uses Hodge theory to construct a relatively ample line bundle. The second uses relative birational geometry to upgrade the Moishezon information. These are supporting theorems here; fibrewise projectivity alone is not substituted for either hypothesis.

For example, $\mathbb{P}^1 \times Y \to Y$ is projective for every compact Kähler Y, with relatively ample bundle from the first factor. If Y is nonprojective, the total space cannot be projective, since it contains Y as a closed submanifold. This example keeps the relative and absolute conclusions separate.

3 Stein localization and descent of sections

A projective morphism to a complex analytic base can be studied on Stein neighborhoods. Proper direct images of coherent sheaves are coherent, and Cartan's theorem A makes nonzero coherent sheaves visible through global sections. This is the analytic replacement for several affine-base arguments in algebraic geometry. The compact-set finiteness condition in Fujino's theorem is what permits repeated shrinking while retaining control near the chosen set.

Lemma 3.1 (Descent of generation)**.** *Let $f : X \to Y$ be a proper surjective morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and let L be a line bundle on Y. Then L is globally generated if and only if f^*L is globally generated. In particular semiampleness descends under such f.*

Proof. The projection formula gives $H^0(X, f^*L) = H^0(Y, L)$. If f^*L is generated, choose x over y and a section nonzero at x. Its corresponding section downstairs is nonzero at y, since its pullback value is nonzero. A line bundle is generated exactly when it has such a section at every point. The converse follows by pulling back generating sections. Apply the argument to a generated positive power for semiampleness. ∎

The hypothesis on $f_*\mathcal{O}_X$ matters: sections on a finite cover need not descend. Likewise, numerical triviality on fibres is not the same as exhibiting a line bundle L with the required pullback identity.

4 How to use the toolkit

For the MRC fibration, descent of holomorphic two-forms supplies the first hypothesis of Theorem 2.1. For a relative Iitaka problem, a relatively big line bundle may supply the second. Once projectivity is established, analytic BCHM gives relative models near appropriate compact subsets. Finally a genuine pullback identity and connected fibres allow section-theoretic conclusions to descend by Lemma 3.1.

The recurring research question is where projectivity is really needed: to construct a model, to apply vanishing, or to descend a positive class. Locating that particular use is more informative than trying to replace every projective theorem simultaneously. None of these reductions alone upgrades a curve-positive transcendental class to a Kähler class.

References

[CH24] Benoît Claudon and Andreas Höring, *Projectivity criteria for kähler morphisms*, 2024.

BDPP and the Positivity-to-Curves Principle

Kähler MMP Yi Li

Contents

1 The projective duality theorem 1

2 The canonical class in the Kähler setting 2

3 The exact consequence needed by the MMP 2

4 Research directions 2

Big picture

This supplement explains the two positivity-to-curves results used by the series. Projective BDPP is a duality theorem for numerical cones; the compact Kähler uniruledness theorem detects the canonical class analytically. Their conclusions meet at the MMP, but their proofs should not be conflated.

1 The projective duality theorem

For a smooth projective variety X, the pseudoeffective cone in $N^1(X)$ is dual to the closed movable cone in $N_1(X)$ [BDPP13]. The easy inclusion follows by restricting effective divisors to general covering curves and passing to limits; this is proved in the rational-curves chapter.

The converse is a separation argument made quantitative. If a divisor lies outside the pseudoeffective cone, one seeks a movable class on which its intersection is negative. Approximate big perturbations on modifications by an ample moving part plus an effective fixed part. The orthogonality estimate says that the latter contributes negligibly in the limiting intersection with powers of the moving part. Pushing those powers down supplies the required movable classes. Fujita approximation by itself does not prove this converse: orthogonality is the additional estimate that makes separation work.

2 The canonical class in the Kähler setting

Ou's theorem states that a smooth compact Kähler manifold is uniruled exactly when its canonical class is not pseudoeffective [Ou25]. Its proof uses analytic positivity and algebraicity criteria for foliations, then a relative Albanese reduction. The algebraicity step is indispensable: a destabilizing analytic direction is not yet a compact rational curve.

The use of projective families is local to particular stages of the argument. It does not mean the original manifold is projective. For example, $\mathbb{P}^1 \times T$ with T a nonprojective compact torus is uniruled and Kähler but nonprojective. Its negative canonical direction is the first factor.

3 The exact consequence needed by the MMP

On a smooth compact Kähler manifold that is not uniruled, the canonical class is pseudo-effective. For projective canonical singularities this extends by the effective discrepancy comparison on a resolution, proved in the rational-curves chapter. This places the space on the minimal-model side of the canonical dichotomy whenever the relevant existence theorem is available.

For a boundary B, the sign of $K_X + B$ must be examined separately. Moreover uniruledness alone does not identify a chosen extremal ray or produce a contraction. The cone and contraction theorems supply those next steps. The longer original BDPP working documents are retained in this folder for subsequent detailed expansion.

4 Research directions

The natural questions are how to extract degree bounds from analytic positivity arguments and how to make them uniform in families. Any proposed cone duality beyond the projective setting must specify the ambient Bott–Chern spaces and which movable objects are used; testing only algebraic curves may lose transcendental information.

References

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.