Preservation of the Kähler Condition

Kähler MMP

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Big picture

The Kähler condition is part of the output of every MMP step. Being bimeromorphic to a Kähler space only places a space in Fujiki's class; it does not furnish a Kähler form on that space. The basic model is a blowup, where a small exceptional correction repairs the degeneracy of a pulled-back form. Contraction reverses the problem and requires a genuine descent and positivity theorem.

1 The local model: repairing a pullback

Proposition 1.1. *Let $\mu : \widetilde{X} \to X$ be the blowup of a smooth compact Kähler manifold along a smooth centre, with exceptional divisor E. If ω is a Kähler form on X, then $\mu^*[\omega] - \varepsilon[E]$ is Kähler for every sufficiently small positive ε.*

Proof. The bundle $\mathcal{O}_E(-E)$ is the relative $\mathcal{O}(1)$ of the projective normal bundle. Choose a smooth metric on $\mathcal{O}_{\widetilde{X}}(-E)$ whose curvature η is positive on the tangent spaces to the fibres of $E \to \mu(E)$. Such a metric is obtained by taking a fibrewise Fubini–Study metric on E and extending it smoothly. Precisely these vertical tangent spaces form the kernel of the semipositive form $\mu^*\omega$ along E.

We use a compact linear-algebra observation. If a continuous semipositive Hermitian form A on a compact vector bundle has a continuous Hermitian form B positive on every nonzero vector in $\ker A$, then $A + \varepsilon B$ is positive for sufficiently small $\varepsilon > 0$. Otherwise there would be unit vectors v_j and $\varepsilon_j \to 0$ with $A(v_j, v_j) + \varepsilon_j B(v_j, v_j) \le 0$. A convergent subsequence would limit to a unit vector v with $A(v, v) = 0$ and $B(v, v) \le 0$, a contradiction.

Apply this observation on the compact manifold $\widetilde{X}$, with a fixed background Hermitian metric for its unit tangent bundle: away from E the kernel is zero. Thus $\mu^*\omega + \varepsilon\eta$ is positive everywhere. It is closed and represents the stated class. ∎

The sign is forced: subtracting E adds positivity in the directions collapsed by the blowup. This argument is the geometric content of the standard blowup lemma; see [Bou02].

2 The contraction theorem needed in dimension three

Theorem 2.1 (Höring–Peternell, Corollary 8.2)**.** *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities, and let R be a K_X-negative extremal ray of the generalized Mori cone. Suppose $f : X \to Y$ is a bimeromorphic contraction, $-K_X$ is f-ample, and the curves contracted by f are precisely those with class in R. Then Y is Kähler [HP16].*

The strategy is to choose a nef supporting class vanishing on R, descend it to the target, and prove that the descended class is Kähler. Descent is cohomological; strict positivity downstairs uses a Kählerness criterion. The theorem is a supporting analytic input to the existence chapter. Generalized threefold variants and their hypotheses are treated in [DHY23].

For a flip one must also establish Kählerness of the flipped space. Projectivity of the flipping morphism is useful here, but projectivity over a target should not be confused with the target being projective. The MMP chapter states the relevant preservation results together with the step-existence theorems.

3 Why curve positivity alone is not a proof

On a nonprojective compact Kähler manifold, the algebraic curves need not test every real $(1,1)$-direction. A descended class can therefore pass all curve tests without the argument having established Kählerness. The smooth Demailly–Păun criterion instead identifies the Kähler cone as a connected component of the cone of classes with positive top intersection on every positive-dimensional irreducible analytic subvariety [DP04]. The component condition also matters. On singular targets one needs the applicable singular criterion, rather than quoting the smooth statement verbatim.

4 Research directions

The useful separation is between descent of a Bott–Chern class and positivity of its descended representative. The blowup calculation suggests how exceptional corrections can repair positivity upstairs; it does not justify arbitrary blowdowns. Higher-dimensional

extensions must explain how the supporting class is controlled on higher-dimensional subvarieties of the target, and how the singularities permit descent of local potentials.

References

[Bou02] S. Boucksom, *On the volume of a line bundle*, International Journal of Mathematics **13** (2002), no. 10, 1043–1063.

[DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.

[DP04] J.-P. Demailly and M. Păun, *Numerical characterization of the kähler cone of a compact kähler manifold*, Annals of mathematics (2004), 1247–1274.

[HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.