A Brief Survery on Kähler MMP Reading Notes Spring 2025

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1 Overview

The aim of this series of notes is to summarize recent developments in the Kähler minimal model program. We will divide the discussion into several topics. This note is intended as an overview of what is currently known about the Kähler minimal model program, and it briefly sketches some important ideas that appear along the way.

Contents

2 Cone Theorems

The original proof of cone theorem requires the Mori bend and break techniques. Unfortunately, Mori bend and break remains an open question for Kähler manifold. Using recent development of BDPP conjecture [Ou25], cone theorem for Kähler varieties indeed holds true.

Theorem 1 ([HP24, Theorem 0.5])**.** Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B+\boldsymbol{\beta})$ is generalized klt (resp. and $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective), then the following holds

(1) (Cone theorem) There are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X \geq 0} + \sum_{i\in I} \mathbb{R}^+ [\Gamma_i]$$

(2) (Length of Extremal rays) that

$$0 < -(K_X + B + \boldsymbol{\beta}_X)\cdot \Gamma_i \leq 2n$$

(3) (Finiteness) if $B+\boldsymbol{\beta}_X$ or $(K_X + B + \boldsymbol{\beta}_X)$ is big, then I is finite.

PROOF IDEA 2.

3 Contraction Theorems

Currently, contraction theorem is only known for Kähler 3-folds. [HP16] proved for $\mathbb{Q}$-factorial compact Käher 3-folds with terminal singularities. [DH25], [DH24a] completely proved the result for Kähler 3-folds $\mathbb{Q}$-factorial dlt pairs.

Theorem 3 ([HP16, Theorem 1.3 + Corollary 7.9 + Theorem 7.12])**.** Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let $\mathbb{R}^+[\Gamma_i]$ be a K_X-negative extremal ray in $\overline{NA}(X)$. Then the contraction of $\mathbb{R}^+[\Gamma_i]$ exists in the Kähler category.

(1) (Divisorial Contraction when $n(\alpha) = 0$) Suppose that the extremal ray R is divisorial and $n(\alpha) = 0$. Then the surface S can be blown down to a point: there exists a bimeromorphic morphism $\varphi : X \to Y$ to a normal compact threefold Y with $\dim \varphi(S) = 0$ such that $\varphi|_{X\setminus S}$ is an isomorphism onto $Y\setminus 0$;

(2) (Divisorial Contraction when $n(\alpha) = 1$) Suppose that the extremal ray R is divisorial and $n(\alpha) = 1$. Then there exists a fibration $f : S \to B$ onto a curve B such that a curve $C \subset S$ is contracted if and only if $[C] \in R$;

(3) (Flipping contraciton case) Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small, and let C be the locus covered by curves in (cf. Notation 7.10). Then C is contractible.

PROOF IDEA 4. We will not give proof details here, but let us sketch the key point of the proof here.

Das-Hacon proved the contraction theorem for compact Kahler $\mathbb{Q}$-factorial dlt pairs.

Theorem 5 ([DH24a, Theorem 1.2])**.** Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler 3-fold dlt pair such that $K_X + B$ is pseudo-effective.

Let ω be a Kähler class such that $\alpha = K_X + B + \omega$ is a nef and big class and $\alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$ is an extremal ray.

Then there exists a projective bimeromorphic morphism $f : X \to Z$ with connected fibers such that $\alpha = f^*\alpha_Z$, where α_Z is a Kähler class on Z.

Moreover, if f is a divisorial contraction, then (Z, f_*B) has $\mathbb{Q}$-factorial klt singularities.

When flipping contraction exist, Das-Hacon proved the following result on existence of klt flips for generalized Kähler pairs in any dimension

Theorem 6 ([DH24a, Theorem 5.12])**.** Let $(X, B + \boldsymbol{\beta})$ be a $\mathbb{Q}$-factorial gdlt pair, where X is a compact analytic variety belonging to Fujiki's class $\mathcal{C}$, and $f : X \to Z$ a flipping contraction. Then the flip of f exists.

4 Base Point Freeness

The transcendental base point freeness is widely open. For transcendental projective pair, the base point free theorem holds true.

Theorem 7 ([DH24b, Theorem 3.5])**.** Let $(X, B + \boldsymbol{\beta})$ be compact Kähler a generalized klt pair such that $B + \boldsymbol{\beta}$ is big. If $K_X + B + \boldsymbol{\beta}_X$ is nef then it is semiample.

5 Existence of minimal models, good minimal models, canonical models

The existence of minimal model is complete known in 3-folds case, and also for 4-folds effective pairs. For higher dimensional setting, the existence of good minimal model also true under maximal Albanese dimension setting.

5.1 Existence of minimal model for Kähler 3-folds

Theorem 8 ([HP16, Theorem 1.1])**.** Theorem 1.1 Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that K_X is pseudoeffective or, equivalently, that X is not uniruled. Then X has a minimal model, i.e. there exists a MMP

$$X \to X'$$

such that $K_{X'}$ is nef.

Theorem 9 ([DH24a, Theorem 1.4])**.** Let (X, B) be a dlt pair where X is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + B$ is pseudo-effective, then there exists a finite sequence of flips and divisorial contractions

$$\phi : X \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_n$$

such that $K_{X_n} + \phi_* B$ is nef.

And later Das-Hacon-Yáñez proved the result for generalized Kähler 3-folds.

Theorem 10 ([DHY23, Theorem 1.2])**.** Let $(X, B+\boldsymbol{\beta})$ be a generalized compact klt Kähler 3-fold pair. (1) If $K_X + B + \boldsymbol{\beta}_X$ is big, then $(X, B+\boldsymbol{\beta})$ has a log terminal model $f : X \dashrightarrow X^{\mathrm{m}}$ and a unique log canonical model $g : X^{\mathrm{m}} \to X^{\mathrm{c}}$.

(2) If $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective and $\boldsymbol{\beta}_X$ is big, then $K_X + B + \boldsymbol{\beta}_X$ has a log terminal model $f : X \dashrightarrow X^{\mathrm{m}}$ and there is a contraction $g : X^{\mathrm{m}} \to Z$ such that $f_*\left(K_X + B + \boldsymbol{\beta}_X\right) \equiv g^*\omega_Z$ where ω_Z is a Kähler form on Z.

5.2 Existence of minimal model for Kähler 4-folds effective pairs

For Kähler folds effective dlt pairs, the existence of minimal model is proved by Das-Hacon-Păun.

Theorem 11 ([DHP24, Theorem 1.1])**.** Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler 4-fold dlt pair such that $K_X +\ B \sim_{\mathbb{Q}} M \geq 0$. Then (X, B) has a log minimal model.

Theorem 12 ([DHP24, Corollary 7.12])**.** Let (X, B) be a $\mathbb{Q}$-factorial compact Kähler plt pair of dimension 4 such that $\kappa\left(X, K_X + B\right) \geq 0$. Then (X, B) has log terminal model.

5.3 Existence of good minimal model for Kähler variety with maximal Albanese dimension

Das-Hacon also prove the existence of good minimal model for Kahler klt pairs with maximal Albanese dimension.

Theorem 13 ([DH24c, Theorem 1])**.** Let (X, B) be a compact Kähler klt pair of maximal Albanese dimension. Then (X, B) has a good minimal model.

PROOF IDEA 14. (The proof idea here is only sketched; more details will be provided in the topic notes.) We first take a log resolution and replace the Albanese map by its projective model $a' : X' \to A$. As the diagram shows below.

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ & \overset{a'}{\searrow} & \downarrow a \\ & & A \end{array}$$

Note that the existence of good minimal model are preserved under the modification $\mu : X' \to X$. We then divide the proof into two cases.

Case 1. When $\kappa(K_X + B) = 0$.

Let D be an irreducible component of the unique effective divisor

$$G \in \left|m(K_X + B)\right|.$$

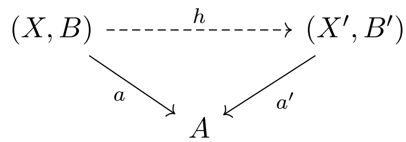
We then prove that the cohomological support locus satisfies

$$V^0(\omega_D) = \emptyset.$$

Using Fourier–Mukai techniques, we show that

$$a_*\omega_D = 0,$$

and therefore D is an a-exceptional divisor. We then run the relative MMP



with

$$K_{X'} + B' \sim_{\mathbb{Q}} E \geq 0,$$

where E is a'-exceptional (its exceptional support being the divisor proved above).

By the negativity lemma, we can then deduce that $E = 0$. In other words,

$$K_{X'} + B' \sim_{\mathbb{Q}} 0.$$

In particular, (X, B) admits a good minimal model.

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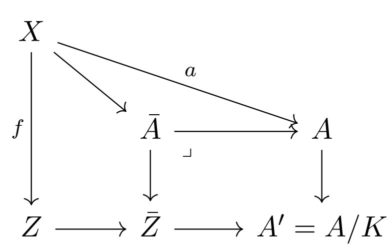
Case 2. When $\kappa(K_X + B) > 0$.

In this case, the proof is more involved. The idea is as follows. We first consider the Iitaka fibration

$$X \dashrightarrow Z,$$

such that the very general fiber F satisfies the conditions of Case 1. Therefore, we can run the relative MMP for F over A, obtaining a complex torus F'. We then have the following commutative diagram (details on its construction will be given in the topic discussion note):



Here, $\bar{A} = A \times_{A'} \bar{Z}$, and the induced map $Z \to \bar{Z} = \operatorname{Proj} R(K_X + B)$, with $K = a(F')$.

We then decompose the log canonical divisor as

$$K_X + B \sim_{\mathbb{Q}} H_X + E,$$

where H_X is semi-ample (as the pullback of some ample divisor $\bar{H}$ on $\bar{Z}$) and $E \geq 0$ is an $\bar{A}$-exceptional divisor. (Details on the construction of this decomposition will be provided in the topic discussion note.)

Next, we run the $(K_X + B + (2n + 3)sH_X)$-MMP over A:

$$\begin{array}{ccc} (X, B + (2n+3)sH_X) & \dashrightarrow & (X', B' + (2n+3)sH'_X) \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

Using the length of extremal rays, we show that the relative MMP above is also an MMP over $\bar{Z}$, and hence over $\bar{A}$. Since $E \geq 0$ is $\bar{A}$-exceptional, the negativity lemma implies that $E' = 0$. Therefore,

$$K_{X'} + B' \sim_{\mathbb{Q}} \mu^* H_{\bar{A}},$$

for some semi-ample divisor $H_{\bar{A}}$ pulled back via $\mu : X' \to \bar{A}$.

Remark 15 (Comparison with the proof for projective varieties)**.** Fujino proved [Fuj13, Theorem 4.3] proved that projective klt pairs with maximal Albanese dimension admits good minimal model. Let us compare Das-Hacon's result with Fujino's result and see what's new in the Kähler setting.

5.4 Existence of good minimal model when general fibers admits good minimal models

Recently, Huang proved the existence of good minimal model for klt Kähler pair, when the Albanese map is projective and general fibers admits good minimal models.

Theorem 16 ([Hua25, Theorem 1])**.** Let $a_X : X \to A$ be the Albanese morphism of a compact Kähler variety X where (X, B) is a klt pair. Assume a_X is projective with connected fibers. Let F be the general fiber of a_X. Suppose (F, B_F) has a good minimal model, then (X, B) has a good minimal model.

6 Rational Curves on Kähler varieties and Ou's BDPP theorem for compact Kähler manifolds

The aim of this section is to give a brief sevey of [Ou25] proof of BDPP conjecture for compact Kähler manifolds.

Theorem 17 ([Ou25, Theorem 1])**.** Let X be a compact Kähler manifold, then K_X is not pseudo-effective iff X is uniruled.

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When singularity is bad (say worse than klt singularities), the BDPP theorem is not true, by the following example of Totaro.

7 Projectivity Criterion for Kähler Varieties

Höring-Claudon developed a projectivity criterion for Kähler morphisms.

Theorem 19 ([CH24, Theorem 1.1])**.** Let $f : X \to Y$ be a fibration between compact Kähler manifolds. Assume one of the following:
(1) The natural map $f^* : H^0\left(Y, \Omega_Y^2\right) \longrightarrow H^0\left(X, \Omega_X^2\right)$ is an isomorphism or
(2) The morphism f is Moishezon, i.e. there exists a line bundle $L \to X$ that is f-big.

Then f is a projective morphism.

PROOF IDEA 20.

8 Canonical Bundle Formulas

Canonical bundle formula for proper morphism between compact Kähler manifold is still an open question. Some partial results are known by [HP24].

Theorem 21 ([HP24])**.**

9 Abundance Conjecture for Kahler varieties

Abundance for Kähler 3-folds lc pairs has been proved by [DO24].

Theorem 22 ([DO24, Theorem 1.1] + [DO25, Corollary 1.3])**.** Let (X, Δ) be a lc pair such that X is a compact Kähler threefold. If $K_X + \Delta$ is nef, then it is semi-ample.

PROOF IDEA 23. Let us stress what's the difference compared with abundance conjecture for projective pair.

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