Projectivity and Kählerness Criteria

Kähler MMP

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1 Introduction

The purpose of these notes is to prove that the Kähler minimal model program preserves the Kähler condition, in arbitrary dimension. We also prove a Kählerness criterion for Fujiki varieties in terms of rational curves, and collect the projectivity criteria that both arguments use.

A bimeromorphic contraction of a compact complex variety need not preserve projectivity, and need not preserve Kählerness. For the contractions arising in the minimal model program both are preserved, and without this the program could not be iterated. In dimension three the statement was known, by Höring–Peternell and by Das–Hacon–Yáñez, through Kleiman's Kählerness criterion, which fails on a general Fujiki variety. The argument given in Section 3 is dimension-free and does not use that criterion.

The setting is the following. For projective pairs the minimal model program is largely settled by the work of Birkar, Cascini, Hacon and McKernan.

Theorem 1.1 (Birkar–Cascini–Hacon–McKernan, [BCHM10])**.** *Let (X, B) be a klt pair with X projective and B big. If $K_X + B$ is pseudo-effective then (X, B) has a good minimal model; otherwise (X, B) has a Mori fibre space.*

Conjecture 1.2 (Kähler analogue)**.** *Let $(X, B + \boldsymbol{\beta})$ be a generalized klt pair with X compact Kähler and $B + \boldsymbol{\beta}_X$ modified big. If $K_X + B + \boldsymbol{\beta}_X$ is pseudo-effective then $(X, B + \boldsymbol{\beta})$ has a good minimal model; otherwise it has a Mori fibre space.*

Three things must be settled before Conjecture 1.2 can be approached: the cone and contraction theorems, which produce the object to be contracted; existence of flips; and the persistence of the hypotheses, that is, whether the conditions imposed on one step of the program still hold for its output. These notes treat the third, for the singularity, $\mathbb{Q}$-factoriality and Kählerness conditions.

Example 1.3. The persistence of Kählerness is not automatic for an arbitrary bimeromorphic contraction. On a generic quintic threefold, blow up a rational curve C_d of degree d whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, so that the exceptional divisor is a product of two rulings, and then contract the second ruling. The resulting variety need be neither Kähler nor projective; it is only Moishezon. What rules this out inside

> the minimal model program is the negativity of the adjoint class on the contracted ray, and that is what the proofs below use.

Big picture

Each of the criteria below has the same form: a positivity property can fail only in the presence of a curve obstructing it, so ruling out that curve certifies the property. Section 3 proves that a flipping or divisorial contraction of a Kähler generalized pair has Kähler target; the difficulty is that the candidate class on the base exists as a Bott–Chern class only after the base is known to have rational singularities, which is precisely what the proof must produce. Section 4 establishes a dichotomy for compact Fujiki varieties that are not Kähler, and Section 5 collects the projectivity criteria of Kodaira, Claudon–Höring and Hacon–Xie.

Contents

2 Two criteria recalled

We shall use the following two results, proved in the companion notes on rational curves. Both are stated for compact varieties in Fujiki's class $\mathcal{C}$, with no Kählerness and no $\mathbb{Q}$-factoriality assumed; this is essential below, where they are applied to a variety not yet known to be Kähler.

Theorem 2.1 (Hacon–Li–Xie, [HLX26, Theorem 3.6]). *Let X be a compact variety in Fujiki's class $\mathcal{C}$ such that $(X, B + \boldsymbol{\beta})$ is generalized klt and $B + \boldsymbol{\beta}_X$ is big. If $\alpha := [K_X + B + \boldsymbol{\beta}_X]$ is nef and NQC but not big, then X is covered by α-trivial rational curves.*

Theorem 2.2 (Hacon–Li–Xie, [HLX26, Theorem 3.7]). *Let X be a compact variety in Fujiki's class $\mathcal{C}$ such that $(X, B + \boldsymbol{\beta})$ is generalized klt. If $\alpha := [K_X + B + \boldsymbol{\beta}_X]$ is nef and big but not Kähler, then there exists an α-trivial rational curve $C \subset X$.*

In contrapositive form these are criteria: the absence of an α-trivial curve forces first bigness and then Kählerness of α.

3 The Kähler MMP preserves the Kähler condition

Theorem 3.1 (Hacon–Li–Xie, [HLX26]). *Let $(X, B + \boldsymbol{\beta})$ be a compact Kähler, generalized klt, strongly $\mathbb{Q}$-factorial generalized pair with $B + \boldsymbol{\beta}_X$ big, and let $f : X \to Z$ be a flipping or divisorial contraction. Then Z is Kähler.*

Idea of proof. Let $\alpha = K_X + B + \boldsymbol{\beta}_X + \omega$ be the nef class supporting the contracted extremal ray R, with ω Kähler. The proof falls into two halves. First one descends both the pair and the class: Stein-locally over Z the transcendental summand $\boldsymbol{\beta}_X + \omega$ is replaced by an $\mathbb{R}$-divisor Δ with $(X, B+\Delta)$ klt and $K_X+B+\Delta \equiv \alpha$, so that $K_X+B+\Delta \equiv_Z 0$; relative base point freeness, now applicable because Δ is a divisor, upgrades this to an $\mathbb{R}$-linear pullback formula, whence f is crepant, Z is klt and has rational singularities, and the Bott–Chern pullback lemma gives $\alpha = f^*\alpha_Z$. Second, one upgrades α_Z from nef to Kähler: were it not Kähler, Theorem 2.2 would produce an α_Z-trivial rational curve on Z, which lifts to an α-trivial curve not contracted by f.

Proof. By hypothesis there is a $(K_X + B + \boldsymbol{\beta}_X)$-negative extremal ray R, supported by a nef class

$$\alpha = K_X + B + \boldsymbol{\beta}_X + \omega, \qquad \omega \text{ Kähler},$$

with $\alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$ and $f = \mathrm{cont}_R$. Here nefness of α is part of the input: the cone and contraction theorems produce the contraction together with its nef supporting class. It would be circular to deduce it from $\alpha = f^*\alpha_Z$, since nefness of α_Z will be derived from that of α.

Step 1: α is big. Put $\boldsymbol{\gamma} := \boldsymbol{\beta} + \bar{\omega}$. Then $(X, B + \boldsymbol{\gamma})$ is generalized klt, $B + \boldsymbol{\gamma}_X$ is big, and $\alpha = K_X + B + \boldsymbol{\gamma}_X$, so Theorem 2.1 applies. If α were not big, X would be covered by a family $\{C_t\}$ of α-trivial rational curves. Each C_t satisfies $[C_t] \in \alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$, hence is contracted by f and contained in $\mathrm{Ex}(f)$, a proper closed subset of X. A covering family cannot lie in a proper closed subset, so α is big. Note that the covering strength of Theorem 2.1 is what is used here: the existence of a single α-trivial curve is no contradiction, since $\mathrm{Ex}(f)$ carries such curves by construction.

Step 2: reduction to a log pair. The assertion is local over Z, so we may assume Z relatively compact Stein. Since f is bimeromorphic it is Moishezon, so there is a log resolution $\nu : X' \to X$ over Z with $X' \to Z$ projective. As $X' \to Z$ is bimeromorphic, $\boldsymbol{\beta}_{X'} + \nu^*\omega$ is nef and big over Z, and hence

$$\boldsymbol{\beta}_{X'} + \nu^*\omega \;\equiv\; \Delta', \qquad (X', B' + \Delta') \text{ sub-klt};$$

the pair upstairs is only sub-klt because the crepant pullback B' may have negative coefficients. Pushing forward, $\boldsymbol{\beta}_X + \omega \equiv \Delta$ with $(X, B + \Delta)$ klt. In particular X has rational singularities.

Step 3: $K_X + B + \Delta \equiv_Z 0$. Since $\Delta \equiv \boldsymbol{\beta}_X + \omega$ we have

$$K_X + B + \Delta \;\equiv\; K_X + B + \boldsymbol{\beta}_X + \omega \;=\; \alpha.$$

The curves contracted by $f = \mathrm{cont}_R$ are exactly those with class in R, and α supports R, so $\alpha \cdot C = 0$ for every such curve; that is, $K_X + B + \Delta \equiv_Z 0$.

> **Remark 3.2.** The contraction f is $(K_X + B + \boldsymbol{\beta}_X)$-negative, so $(K_X + B + \boldsymbol{\beta}_X) \cdot C < 0$ for contracted curves, and it is the identity
>
> $$(K_X + B + \Delta) \cdot C \;=\; (K_X + B + \boldsymbol{\beta}_X) \cdot C + \omega \cdot C$$
>
> together with the fact that the two terms cancel on R that gives Step 3. It is therefore essential that Δ absorb the Kähler class ω and not merely the moduli part $\boldsymbol{\beta}_X$: otherwise $K_X + B + \Delta$ would be f-negative, and the log canonical model of Step 4 would not be Z.

Step 4: the log canonical model of $(X, B + \Delta)$ over Z is Z. Apply the relative base point free theorem to $f : X \to Z$ and $H := K_X + B + \Delta$. Its hypotheses hold: $(X, B + \Delta)$ is klt by Step 2; H is f-nef, indeed $H \equiv_Z 0$ by Step 3; and $H - (K_X + B + \Delta) = 0$ is f-nef and f-big, the latter since f is bimeromorphic. We obtain a factorisation

$$f : X \xrightarrow{\varphi} W \xrightarrow{g} Z, \qquad g \text{ projective}, \quad \varphi^* A = K_X + B + \Delta,$$

with A a g-ample $\mathbb{R}$-Cartier class on W. Now φ contracts precisely those f-vertical curves C with $(K_X + B + \Delta) \cdot C = 0$, which by Step 3 is every f-contracted curve. Hence g is a bimeromorphic morphism contracting no curve, so it is finite; as Z is normal, $W = Z$ and $\varphi = f$.

Step 5: f is crepant and Z has rational singularities. By Step 4, $K_X + B + \Delta = f^*A$ with A an $\mathbb{R}$-Cartier class on Z. Since f is bimeromorphic, the projection formula gives

$f_*f^*A = A$, while $f_*(K_X + B + \Delta) = K_Z + B_Z + \Delta_Z$ with $B_Z := f_*B$ and $\Delta_Z := f_*\Delta$. Therefore $A = K_Z + B_Z + \Delta_Z$ is $\mathbb{R}$-Cartier and

$$K_X + B + \Delta \ \sim_{\mathbb{R}} \ f^*(K_Z + B_Z + \Delta_Z). \tag{1}$$

Thus f is crepant, $(Z, B_Z + \Delta_Z)$ is klt, and Z has rational singularities.

Remark 3.3. Numerical triviality over Z alone would not suffice. The relation $\equiv_Z 0$ carries no information about $K_Z + B_Z + \Delta_Z$ being $\mathbb{R}$-Cartier, and without $\mathbb{R}$-Cartierness there is no pair on Z, hence no discrepancies, no kltness and no rational singularities. The passage from $\equiv_Z 0$ to (1) is exactly what base point freeness supplies, and it is available only because Δ is a divisor. In the divisorial case one might expect a discrepancy term $+aE$ in (1); its absence is the content of crepancy, and combined with kltness of $(X, B + \Delta)$ it gives $a(E; Z, B_Z + \Delta_Z) = -\operatorname{coeff}_E(B + \Delta) > -1$.

Step 6: α descends. Recall the pullback lemma for Bott–Chern classes: if $f : X \to Z$ is a proper bimeromorphic morphism with connected fibres between normal compact complex spaces in Fujiki's class $\mathcal{C}$ with rational singularities, then $f^* : H^{1,1}_{\mathrm{BC}}(Z) \to H^{1,1}_{\mathrm{BC}}(X)$ is injective with image

$$\{\eta \ : \ \eta \cdot C = 0 \ \text{ for every curve } C \text{ with } f_*C = 0\}.$$

All hypotheses are available: X is Kähler and Z is its bimeromorphic image, so both lie in class $\mathcal{C}$, while Z is not yet known to be Kähler; X has rational singularities by Step 2 and Z by Step 5; and α is trivial on every f-contracted curve by Step 3. Hence there is a unique $\alpha_Z \in H^{1,1}_{\mathrm{BC}}(Z)$ with $\alpha = f^*\alpha_Z$, and pushing forward, $\alpha_Z = K_Z + B_Z + \boldsymbol{\beta}_Z + f_*\omega$. Applying crepancy to the generalized pair, $(Z, B_Z + \boldsymbol{\beta}_Z + f_*\omega)$ is generalized klt.

Remark 3.4. Rational singularities of Z is not a technical convenience here. The proof of the pullback lemma runs the exponential-type sequence $0 \to \mathbb{R} \to \mathcal{O} \to \mathcal{H} \to 0$ together with a Leray argument, and injectivity uses $H^1(Z, \mathcal{O}_Z) \cong H^1(\hat{Z}, \mathcal{O}_{\hat{Z}})$ for a resolution $\hat{Z} \to Z$, which is the definition of rational singularities. This is why Step 5 must precede Step 6. Note also that the correct b-class for the generalized pair is $\boldsymbol{\beta} + \bar{\omega}$, taken from X, rather than $\overline{f_*\omega}$ taken from Z: the latter would be b-nef only if $f_*\omega$ were nef on Z, whereas $f_*\omega$ is big and not nef.

Step 7: α_Z is nef and big. Let $C \subset Z$ be an irreducible curve and choose an irreducible $C' \subset X$ with $f(C') = C$, say of degree $m > 0$ over C. By the projection formula,

$$\alpha_Z \cdot C \ = \ \tfrac{1}{m}\,\alpha_Z \cdot f_*C' \ = \ \tfrac{1}{m}\, f^*\alpha_Z \cdot C' \ = \ \tfrac{1}{m}\,\alpha \cdot C' \ \geq \ 0,$$

so α_Z is nef; surjectivity of f is what makes this a descent, rather than the trivial statement that a pullback of a nef class is nef. Bigness is preserved by pushforward under a proper bimeromorphic morphism, so $\alpha_Z = f_*\alpha$ is big by Step 1.

Step 8: no curve on Z is α_Z-trivial. Suppose $\alpha_Z \cdot C = 0$ for some irreducible curve $C \subset Z$, and choose C' as in Step 7. Then

$$0 = m(\alpha_Z \cdot C) = \alpha_Z \cdot f_* C' = f^*\alpha_Z \cdot C' = \alpha \cdot C',$$

so C' is α-trivial, hence $[C'] \in R$ and f contracts C', contradicting $f(C') = C$.

Step 9: conclusion. By Steps 6 and 7 the class α_Z is the adjoint class of the generalized klt pair $(Z, B_Z + \boldsymbol{\beta}_Z + f_*\omega)$ on the compact Fujiki variety Z, and it is nef and big. If it were not Kähler, Theorem 2.2 would give an α_Z-trivial rational curve on Z, contradicting Step 8. Hence α_Z is Kähler and Z is Kähler. ■

Remark 3.5. Both dichotomies invoked above, in Step 1 and in Step 9, are closed by the same observation: f has already contracted every α-trivial curve. The substance of the argument therefore lies in Steps 2–6, which manufacture on Z the hypotheses of Theorem 2.1 and Theorem 2.2; the two theorems themselves are full minimal model program arguments carried out on a Fujiki, rather than Kähler, variety.

4 A Kählerness criterion via rational curves

We turn to the following dichotomy for compact Fujiki varieties, an analogue of which for Moishezon spaces is due to Villalobos-Paz.

Conjecture 4.1 (Hacon–Li–Xie, [HLX26])**.** *Let $(X, B + \boldsymbol{\beta})$ be a compact generalized klt pair in Fujiki's class $\mathcal{C}$. If X is not Kähler, then either*

1. *X contains a rational curve C with $-[C] \in \overline{\mathrm{NA}}(X)$, or*
2. *there exists a small $\mathbb{Q}$-factorial Kähler modification $\mu : X^{\mathrm{qf}} \to X$.*

Notation 4.2. Throughout this section $\nu : X' \to X$ denotes a Kähler resolution with Kähler class ω', and $D := K_{X'} + B' + \boldsymbol{\beta}_{X'}$. We write

$$t := \inf\{\tau \geq 0 \mid D + \tau\,\omega' \text{ is nef over } X\}$$

for the **relative nef threshold**, $\alpha := D + t\,\omega'$, and $\psi : X' \to \bar{X}$ for the log canonical model of α over X. Let $F \subset \overline{\mathrm{NA}}(X'/X)$ denote the face spanned by the ψ-contracted curves; for very general ω' it is a single ray $\mathbb{R}^+[C]$.

Idea of proof. If $t = 0$ then the relative negative part of D vanishes, the resolution is small, and (2) holds. If $t > 0$ one forms the log canonical model at the threshold, for which no minimal model program is needed since α is nef and big over X, hence relatively semiample. The heart of the argument is then Lemma 4.3: the contracted ray either releases a rational curve realising (1), or is extremal in the *absolute* Mori cone. In the latter case Theorem 3.1 makes $\bar{X}$ Kähler and the threshold drops strictly; finiteness of models forces $t = 0$ after finitely many steps.

Lemma 4.3. *In the situation of Notation 4.2, assume $t > 0$ and let $\mathbb{R}^+[C] = F$. Then either X contains a rational curve R with $-[R] \in \overline{\mathrm{NA}}(X)$, or $\mathbb{R}^+[C]$ is a D-negative extremal ray of the absolute cone $\overline{\mathrm{NA}}(X')$.*

Proof. Assume $\mathbb{R}^+[C]$ is not extremal in $\overline{\mathrm{NA}}(X')$. For $0 < \delta < t$ set

$$\alpha_\delta \ := \ D + (t-\delta)\,\omega' \ = \ \alpha - \delta\,\omega'.$$

This is again the adjoint class of a generalized klt Kähler pair on X', with $(t-\delta)\omega'$ playing the role of the Kähler summand, so the cone theorem gives

$$\overline{\mathrm{NA}}(X') \ = \ \overline{\mathrm{NA}}(X')_{\alpha_\delta \geq 0} \ + \sum_{R \in S_\delta} R, \qquad S_\delta := \{\text{extremal rays } R \ : \ \alpha_\delta \cdot R < 0\},$$

with S_δ finite and each of its members spanned by a rational curve.
If $z \in \overline{\mathrm{NA}}(X') \setminus \{0\}$ satisfies $\alpha_\delta \cdot z \geq 0$, then

$$\alpha \cdot z \ = \ \alpha_\delta \cdot z + \delta\,\omega' \cdot z \ \geq \ \delta\,\omega' \cdot z \ > \ 0,$$

since a Kähler class is strictly positive on $\overline{\mathrm{NA}}(X') \setminus \{0\}$. Moreover for $\delta' < \delta$ we have $\alpha_\delta = \alpha_{\delta'} - (\delta - \delta')\omega'$, so $\alpha_{\delta'} \cdot R < 0$ implies $\alpha_\delta \cdot R < 0$: the finite sets S_δ shrink as δ decreases, hence stabilise, with

$$\bigcap_{\delta > 0} S_\delta \ = \ \{R \ : \ \alpha \cdot R \leq 0\}.$$

Combining, for all sufficiently small δ we may write

$$[C] \ = \ \sum_{i=1}^{k} \sigma_i [\Sigma_i] + \gamma, \qquad \sigma_i > 0, \tag{2}$$

where each Σ_i is a rational curve spanning an extremal ray of $\overline{\mathrm{NA}}(X')$ with $\alpha \cdot \Sigma_i \leq 0$, and γ is either 0 or satisfies $\alpha \cdot \gamma > 0$. Each such Σ_i is automatically D-negative, since $D \cdot \Sigma_i = \alpha \cdot \Sigma_i - t\,\omega' \cdot \Sigma_i < 0$; this is why the cone theorem detects it and why it may be taken rational.
We claim $k \geq 1$. Indeed $[C] \neq 0$, so if the sum in (2) were empty we would have $[C] = \gamma \neq 0$ and hence $\alpha \cdot C > 0$; but $\alpha \cdot C = \psi^* \bar{\alpha} \cdot C = \bar{\alpha} \cdot \psi_* C = 0$.
Suppose some Σ_i is not ν-vertical. Since C is ψ-contracted it is ν-vertical, so $\nu_*[C] = 0$, and pushing (2) forward gives $0 = \sum_j \sigma_j [\nu_* \Sigma_j] + \nu_* \gamma$. Solving for the i-th term,

$$-[\nu_* \Sigma_i] \ = \ \sum_{j \neq i} \frac{\sigma_j}{\sigma_i} [\nu_* \Sigma_j] + \frac{1}{\sigma_i}\,\nu_* \gamma \ \in \ \overline{\mathrm{NA}}(X),$$

because ν_* maps $\overline{\mathrm{NA}}(X')$ into $\overline{\mathrm{NA}}(X)$. As Σ_i is rational and not contracted, $R := \nu_*\Sigma_i$ is a rational curve on X with $-[R] \in \overline{\mathrm{NA}}(X)$, which is the first alternative.
Otherwise every Σ_i is ν-vertical, so $[\Sigma_i] \in \overline{\mathrm{NA}}(X'/X)$. Since α is nef over X we get $\alpha\cdot\Sigma_i \geq 0$, and with $\alpha\cdot\Sigma_i \leq 0$ from (2) this forces $\alpha\cdot\Sigma_i = 0$. Hence $[\Sigma_i] \in \alpha^\perp \cap \overline{\mathrm{NA}}(X'/X) = F = \mathbb{R}^+[C]$, so $\mathbb{R}^+[\Sigma_1] = \mathbb{R}^+[C]$; but $\mathbb{R}^+[\Sigma_1]$ is extremal in $\overline{\mathrm{NA}}(X')$, contrary to assumption. ∎

Remark 4.4. The perturbation $\alpha \rightsquigarrow \alpha_\delta$ serves one purpose. Applied to α itself, the cone theorem produces only rays on which α is strictly negative, so the sum would be empty precisely when α is nef, and the conclusion would be vacuous in the case of interest. Passing to α_δ converts the condition "α-trivial or α-negative" into the single condition "α_δ-negative", which the cone theorem detects, and finiteness of the S_δ makes the limit $\delta \to 0$ harmless. Nefness of α over X enters only once, in the last paragraph.

Proposition 4.5. *In the situation of Notation 4.2 with $t > 0$, suppose $\mathbb{R}^+[C]$ is extremal in $\overline{\mathrm{NA}}(X')$. Then either X contains a curve R with $-[R] \in \overline{\mathrm{NA}}(X)$, or $\bar{X}$ is Kähler.*

Proof. Write $\Gamma := \mathbb{R}^+[C]$. The supporting class to be used is not $\alpha = D + t\,\omega'$, which was never assumed absolutely nef, but the nef class α_Γ furnished by the existence theorem for nef supporting classes of a negative extremal ray: $\alpha_\Gamma - D$ is Kähler and $\alpha_\Gamma^\perp \cap \overline{\mathrm{NA}}(X') = \Gamma$. That theorem is proved by a compactness argument on the unit sphere of $N_1(X')$ which never refers to the target of the contraction, so it may be applied before $\bar{X}$ is known to be Kähler. A curve on X' is α_Γ-trivial exactly when its class lies in Γ, so cont_Γ contracts precisely the α_Γ-trivial curves, and Theorem 3.1 applies: the target of cont_Γ is Kähler.

It remains to identify cont_Γ with ψ. Both contract Γ, so cont_Γ factors as $X' \xrightarrow{\psi} \bar{X} \xrightarrow{h} Y$, and the two coincide if and only if every curve with class in Γ is ν-vertical. Let Σ be a curve with $[\Sigma] = \lambda[C]$, $\lambda > 0$. If Σ is ν-vertical then $[\Sigma] \in \alpha^\perp \cap \overline{\mathrm{NA}}(X'/X) = F$, so Σ is ψ-contracted. If not, then $\nu_*\Sigma$ is a curve on X with $[\nu_*\Sigma] = \lambda\,[\nu_*C] = 0$, so $-[\nu_*\Sigma] = 0 \in \overline{\mathrm{NA}}(X)$ and the first alternative holds in the strong form of a numerically trivial curve. In the remaining case h contracts no curve and is therefore an isomorphism, so $\bar{X}$ is Kähler. ∎

Remark 4.6. The coincidence of the two rays does not by itself identify the two contractions. That inference is valid whenever being contracted is detected by the numerical class, which is what an absolutely positive supporting class provides and is why the point is automatic in the projective setting. Here the supporting class of ψ is Kähler only over X, so on X' it detects only the ν-vertical curves, and it is the dichotomy above that breaks the circle.
One point remains open. Conclusion (1) of Conjecture 4.1 asks for a rational curve, whereas Proposition 4.5 produces a rational curve only when the escaping Σ may be taken rational. Closing this requires rational chain connectedness of the fibres of a

> negative extremal contraction in the compact Kähler generalized klt setting; failing that, conclusion (1) must be stated with "curve" in place of "rational curve".

Proof of Conjecture 4.1. *Step 1: normalisation of the resolution.* As X is Fujiki, choose a Kähler resolution $\nu : X' \to X$ with Kähler class ω' as in Notation 4.2, arranged so that $\operatorname{Supp} N(D/X) = \operatorname{Ex}(\nu)$. Writing

$$K_{X'} + B' + \boldsymbol{\beta}_{X'} \;=\; \nu^*(K_X + B + \boldsymbol{\beta}_X) + G, \qquad B' \;=\; \nu_*^{-1}(B) + (1-\epsilon)\operatorname{Ex}(\nu)$$

for $0 < \epsilon \ll 1$, the pair upstairs is klt, as required by the minimal model program arguments below, and $G \geq 0$ has support exactly $\operatorname{Ex}(\nu)$.
Step 2: the case $t = 0$. Here D is nef over X, so $N(D/X) = 0$; by Step 1 the support of the relative negative part is $\operatorname{Ex}(\nu)$, so ν contracts no divisor and is small. Conclusion (2) then follows from Step 6 below.
Step 3: the case $t > 0$. The class $\alpha = D + t\,\omega'$ is nef over X by definition of t, and big over X since ν is bimeromorphic; hence it is semiample over X and the log canonical model $\psi : X' \to \bar{X}$ over X exists without recourse to a minimal model program, with $K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} + t\,\bar{\omega}$ Kähler over X. For very general ω' the face F is a single ray $\mathbb{R}^+[C]$ with $[C] \neq 0$. By Lemma 4.3 and Proposition 4.5 we may assume conclusion (1) fails, so that $\bar{X}$ is Kähler.
Step 4: the flip or the divisorial contraction. Let $X^+ \to \bar{X}$ be the log canonical model of D over $\bar{X}$, which exists because $-D$ is Kähler over $\bar{X}$. If ψ is divisorial then $X^+ = \bar{X}$; if ψ is small then $X^+ \to \bar{X}$ is the flip of ψ. Taking the model of D rather than of $D + (t-\epsilon)\omega'$ is harmless: over $\bar{X}$ one has $D \equiv_{\bar{X}} -t\,\omega'$ and $D + (t-\epsilon)\omega' \equiv_{\bar{X}} -\epsilon\,\omega'$, which are positive multiples of one another, and log canonical models depend on the relative class only up to positive scaling.
Step 5: the threshold drops. From $\psi^*(K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} + t\,\bar{\omega}) = D + t\,\omega'$ one obtains the corresponding identity on X^+, and hence, for $0 < t - s \ll 1$,

$$K_{X^+} + B^+ + \boldsymbol{\beta}_{X^+} + s\,\omega^+ \;=\; f^*\big(K_{\bar{X}} + \bar{B} + \boldsymbol{\beta}_{\bar{X}} + t\,\bar{\omega}\big) - (t-s)\,\omega^+,$$

the sum of the pullback of an absolutely Kähler class and an f-Kähler class, hence absolutely Kähler. Kählerness over X would yield only the inequality $t^+ \leq s < t$; absolute Kählerness yields this and in addition keeps X^+ Kähler, which is what permits the argument to be repeated. Replacing $(X', B' + \boldsymbol{\beta}_{X'} + \omega')$ by $(X^+, B^+ + \boldsymbol{\beta}_{X^+} + s\,\omega^+)$ and iterating produces a strictly decreasing sequence of thresholds $t_0 > t_1 > t_2 > \cdots$ with pairwise distinct log canonical models.
Step 6: termination and conclusion. Termination is checked locally over X: cover X by relatively compact Stein open sets, realise ω' there by an ample $\mathbb{R}$-divisor, trade the moduli part for a boundary, and apply finiteness of weak log canonical models. Hence $t_n = 0$ for some n, and $X_n \to X$ is small by Step 2. Let $\mu : X^{\mathrm{qf}} \to X_n$ be a small strongly $\mathbb{Q}$-factorial modification. Since μ is projective and X_n is Kähler, the class $\mu^*(\text{Kähler}) + (\mu\text{-ample})$ is Kähler on X^{qf}. The composite $X^{\mathrm{qf}} \to X_n \to X$ is therefore small, strongly $\mathbb{Q}$-factorial and Kähler, which is conclusion (2). ■

5 Projectivity criteria

In this section X denotes a normal compact Kähler space, $f : X \to Y$ a fibration, $\Omega^{[2]}$ the sheaf of reflexive 2-forms, and $d := \dim X - \dim Y$.

5.1 Kodaira's criterion

Proposition 5.1 ([CH24, Proposition 2.1])**.** *Let X be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$. Then X is projective.*

Under Hodge symmetry the hypothesis reads $H^0(X, \Omega^2_X) = 0$.

Proof. Suppose first that X is a compact Kähler manifold. Since $H^2(X, \mathcal{O}_X) = 0$, the Hodge decomposition collapses to $H^2(X, \mathbb{R}) = H^{1,1}_{\mathrm{BC}}(X)$, so rational classes are dense among the real $(1,1)$-classes. The Kähler cone is open, hence contains a rational class, which by the exponential sequence

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{R}) \longrightarrow H^2(X, \mathcal{O}_X) = 0$$

lifts to a $\mathbb{Q}$-line bundle; this bundle is ample by the Kodaira embedding theorem. In the singular case, pass to a resolution: once the resolution is projective, X is a Kähler Moishezon variety with rational singularities, hence projective. ■

Remark 5.2. The vanishing hypothesis cannot be dropped. In general $H^{1,1}_{\mathrm{BC}}(X)$ is a proper subspace of $H^2(X, \mathbb{R})$ and may contain no nonzero rational class at all, much as an irrational line in the plane meets the rational lattice only at the origin. The converse also fails: a projective K3 surface has $H^2(X, \mathcal{O}_X) \neq 0$.

5.2 The relative criterion of Claudon and Höring

Theorem 5.3 ([CH24, Theorem 1.1])**.** *Let $f : X \to Y$ be a fibration between compact Kähler manifolds. Assume that either*

(a.1) $f^* : H^0(Y, \Omega^2_Y) \to H^0(X, \Omega^2_X)$ is an isomorphism, or

(a.2) f is Moishezon, that is, there is a line bundle L on X which is f-big.

Then f is a projective morphism.

Theorem 5.4 ([CH24, Theorem 3.1])**.** *Let $f : X \to Y$ be a fibration between normal compact Kähler spaces, with X having strongly $\mathbb{Q}$-factorial klt singularities. Assume that either*

(a.1) Y has klt singularities and $f^* : H^0(Y, \Omega^{[2]}_Y) \to H^0(X, \Omega^{[2]}_X)$ is an isomorphism, or

(a.2) f is Moishezon.

Then f is a projective morphism.

Idea of proof. Case (a.1) for smooth X and Y is a direct construction: one takes a rational class near a Kähler one, uses (a.1) to see that its $(2,0)$-part is a pullback, subtracts a rational pullback obtained from a pushforward formula, and is left with a rational $(1,1)$-class that is still relatively Kähler. Case (a.2) resolves f projectively and descends a relative minimal model program, checking that projectivity over Y survives each step. The singular statement is reduced to these two.

Proof of Theorem 5.3, case (a.1). Fix a Kähler class ω on X. Since Y is compact and relative Kählerness is an open condition, we may choose a rational class $\eta \in H^2(X,\mathbb{Q})$ close enough to ω that its $(1,1)$-part remains relatively Kähler. Hypothesis (a.1) gives the Hodge decomposition

$$\eta \;=\; f^*(\gamma) + \beta + f^*(\bar{\gamma}), \qquad \gamma \in H^{2,0}(Y), \tag{3}$$

with β of type $(1,1)$; restricting to a fibre kills the pullback terms, so $\eta|_{X_y} = \beta|_{X_y}$.
The pushforward f_* is a morphism of Hodge structures of bidegree $(-d,-d)$, so it annihilates classes of degree less than $2d$. Expanding η^d by (3) and applying the projection formula, every term with at least one pullback factor contributes $f_*(\beta^{d-j})$ in negative degree, whence $f_*(\eta^d) = f_*(\beta^d)$. For η^{d+1} only the terms with $j = 0, 1$ survive:

$$f_*(\eta^{d+1}) \;=\; f_*(\beta^{d+1}) + (d+1)\, f_*(\eta^d)\,(\gamma + \bar{\gamma}). \tag{4}$$

This is the Hodge decomposition of $f_*(\eta^{d+1})$ in $H^2(Y)$: the class β^{d+1} has type $(d{+}1, d{+}1)$, so its pushforward is of type $(1,1)$, while $\beta^d \wedge \gamma$ has type $(d{+}2, d)$, so its pushforward is of type $(2,0)$, and symmetrically for $\bar{\gamma}$.
Since f_* is defined over $\mathbb{Q}$ and η is rational, $f_*(\eta^d) = \int_F \beta^d$ is a positive rational number, positive because β restricts to a Kähler class on the general fibre F. Set

$$\eta_Y \;:=\; \frac{1}{(d+1)\, f_*(\eta^d)}\, f_*\Big(\eta^{d+1}\Big) \in H^2(Y,\mathbb{Q}), \qquad \tilde{\eta} \;:=\; \eta - f^*(\eta_Y).$$

Substituting (4), the term in $\gamma + \bar{\gamma}$ cancels the corresponding term of (3), leaving

$$\tilde{\eta} \;=\; \beta - \frac{1}{(d+1) f_*(\eta^d)}\, f^*\big(f_*(\beta^{d+1})\big),$$

a rational class of type $(1,1)$ with $\tilde{\eta} - \eta \in f^* H^2(Y,\mathbb{Q})$.
By the Lefschetz theorem on $(1,1)$-classes, $m\tilde{\eta} = c_1(L)$ for some $L \in \mathrm{Pic}(X)$ and m sufficiently divisible. As $\tilde{\eta}$ differs from η by a pullback and β is relatively Kähler, $c_1(L)$ is relatively Kähler; by Fujiki's lemma there is a Kähler class $[\omega_Y]$ on Y with $c_1(L) + f^*[\omega_Y]$ Kähler, and the relative ampleness criterion then shows that L is f-ample. Hence f is projective. ■

Proof of Theorem 5.3, case (a.2). By Chow's lemma there is a log resolution $\mu_0 : X_0 \to X$ with both μ_0 and $f \circ \mu_0$ projective. Decompose the bimeromorphic map μ_0 into a sequence of divisorial contractions and flips over X, and write $\mu_i : X_i \to X$ and $f_i := f \circ \mu_i$. Then f_0 is projective and $f_m \simeq f$, so it suffices to show that projectivity over Y is preserved by a single step.
Let $\varphi : X_0 \to Z$ be the elementary Mori contraction of an extremal ray $\mathbb{R}^+[C] \subset \overline{\mathrm{NE}}(X_0/X)$. We claim this ray remains extremal in the larger cone $\overline{\mathrm{NE}}(X_0/Y)$. Let $l_1, l_2 \in \overline{\mathrm{NE}}(X_0/Y)$ with $l_1 + l_2 \in \mathbb{R}^+[C]$. Applying φ_*, which annihilates $[C]$, gives $\varphi_* l_1 + \varphi_* l_2 = 0$ in $\overline{\mathrm{NE}}(Z)$. Since Z is Kähler, a Kähler class is strictly positive on $\overline{\mathrm{NE}}(Z) \setminus \{0\}$, so this cone is pointed and $\varphi_* l_1 = \varphi_* l_2 = 0$; as $\ker \varphi_* = \mathbb{R}[C]$ we conclude $l_j \in \mathbb{R}^+[C]$, proving the claim.
The fundamental theorems now provide a projective contraction $\eta : X_0 \to \tilde{X}_1$ with $\tilde{X}_1 \to Y$ projective. Since φ and η contract the same curves, the rigidity lemma gives an isomorphism $\tilde{X}_1 \xrightarrow{\sim} Z$ identifying them. If φ is divisorial then $Z \simeq X_1$ and $X_1 \to Y$ is projective; if φ is small then the flip $\varphi^+ : X_1 \to Z$ is projective, being polarised by K_{X_1}, which is relatively ample over Z, so again $X_1 \to Y$ is projective. The result follows by induction on the number of steps. ∎

Proof of Theorem 5.4. Let $\mu_Y : Y' \to Y$ and $\mu_X : X' \to X$ be projective modifications by compact Kähler manifolds inducing a fibration $f' : X' \to Y'$. Reflexive forms are invariant under resolution, so

$$H^0(Y', \Omega^2_{Y'}) \simeq H^0(Y, \Omega^{[2]}_Y), \qquad H^0(X', \Omega^2_{X'}) \simeq H^0(X, \Omega^{[2]}_X),$$

and the always-injective map $(f')^*$ inherits surjectivity from the hypothesis, hence is an isomorphism. By case (a.1), f' is projective. Consequently f is Moishezon: pushing a relatively ample line bundle on X' down to X yields a relatively big line bundle, the pushforward being a line bundle because X is strongly $\mathbb{Q}$-factorial. Case (a.2) now applies. ∎

Remark 5.5. Taking Y to be a point in (a.1) recovers Proposition 5.1: the source of the map is zero, so the hypothesis reads $H^0(X, \Omega^2_X) = 0$. Injectivity of f^* is automatic for a fibration, so the content of the hypothesis is surjectivity, that is, that every holomorphic 2-form on X descends. The two alternatives are relative forms of two different classical statements: (a.1) of Kodaira's criterion, (a.2) of the fact that a Moishezon Kähler variety is projective. This also explains why the singular version requires Y klt in (a.1) but not in (a.2): reflexive 2-forms behave like forms on a resolution only under klt hypotheses, and (a.1) alone is Hodge-theoretic.

5.3 The variant of Hacon and Xie

The following criterion concerns a Moishezon morphism from a Kähler variety with strongly $\mathbb{Q}$-factorial klt singularities, and imposes no Kähler hypothesis on the target.

Lemma 5.6 (Hacon–Xie, [HX26, Lemma 2.40])**.** *Let $f : X \to Z$ be a proper morphism between compact complex spaces, with X a strongly $\mathbb{Q}$-factorial klt Kähler variety. Suppose there is a projective bimeromorphic morphism $h : W \to X$ with W a Kähler manifold such that $W \to X \to Z$ is projective. Then f is projective. In particular, a strongly Moishezon f is projective in this setting.*

Proof. Put $X_0 := W$ and $\mu_0 := h$, so that μ_0 and $f \circ \mu_0$ are projective. For $0 < \epsilon \ll 1$ write

$$K_{X_0} + (1-\epsilon)\,\mathrm{Ex}(\mu_0) \;=\; \mu_0^* K_X + F,$$

with $F \geq 0$ exceptional and $\mathrm{Supp}(F) = \mathrm{Ex}(\mu_0)$, so that a minimal model program for this boundary over X contracts only μ_0-exceptional curves and terminates at X. Running it gives

$$W = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X,$$

with each X_i projective over the Kähler variety X, hence Kähler. Let $\varphi : X_i \to Y_i$ be the extremal contraction of the i-th step, so that Y_i is Kähler as well.

We first check that the contracted ray $\mathbb{R}^+[C_i]$, extremal in $\overline{\mathrm{NE}}(X_i/X)$, remains extremal in $\overline{\mathrm{NE}}(X_i/Z)$. If not, write $[C_i] = l_1 + l_2$ with l_1, l_2 linearly independent in $\overline{\mathrm{NE}}(X_i/Z)$. Pushing forward gives $\varphi_* l_1 + \varphi_* l_2 = 0$ in $\overline{\mathrm{NE}}(Y_i)$, a pointed cone since Y_i is Kähler, so $\varphi_* l_1 = \varphi_* l_2 = 0$; but then l_1, l_2 lie in $\overline{\mathrm{NE}}(X_i/X)$, contradicting extremality there.

Now suppose $X_i \to Z$ is projective. By the previous paragraph φ contracts an extremal ray of $\overline{\mathrm{NE}}(X_i/Z)$ on the Kähler variety X_i, so the relative cone theorem over Z yields a contraction $c_i : X_i \to Y_i'$ with $Y_i' \to Z$ projective, characterised by contracting exactly the curves with class in $\mathbb{R}^+[C_i]$. Both φ and c_i are proper with connected fibres onto normal spaces and contract the same curves, so by rigidity each factors through the other; the induced map $Y_i \to Y_i'$ contracts no curve and has connected fibres, hence is finite and bimeromorphic onto a normal space, hence an isomorphism. Thus $c_i = \varphi$ and $Y_i \to Z$ is projective, whence $X_{i+1} \to Z$ is projective: in the divisorial case $X_{i+1} = Y_i$, and in the flipping case $X_{i+1} \to Y_i$ is the flip, projective over Y_i. Since $X_0 \to Z$ is projective by hypothesis, induction gives that $X_n = X \to Z$ is projective. ∎

Remark 5.7. This is the form in which the Kähler minimal model program applies the criterion, a log resolution being what is available in practice. The hypothesis that W be Kähler, and not merely Moishezon, is used exactly once, to force every Y_i to be Kähler; the argument on extremality rests on the Mori cone of Y_i being pointed, which can fail for a Moishezon variety.

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