Rational curves on analytic space reading note Spring 2025

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1 Overview

In this note, we will study the rational curves on Moishezon space and Kähler space. The major references of this note are [CH20] and [HP16].

2 Höring-Peternell's approach of Mori bend and break for Kähler 3-fold

In this section, we will briefly summarize the idea of Höring-Peternell on Mori bend and break for Kähler 3-fold [HP16].

Theorem 1. Let X be a normal $\mathbb{Q}$-factorial compact Kähler 3-fold with terminal singularity, with K_X being pseudo-effective.

1. If C is a curve with large anti-canonical degree, then it can break into
$$[C] = [C_1] + [C_2]$$
2. Let $\overline{\mathrm{NE}(X)}$ has a K_X-negative extremal ray $\mathbb{R}_+[\Gamma]$, such that the representative Γ is not very rigid. Then we can find a representable $C \in \mathbb{R}_+[\Gamma]$ such that $\dim_C \mathrm{Chow}(X) > 0$, and $\mathbb{R}_+[\Gamma]$ contains ratioal curves.

Remark 2. Let us first briefly sketch the idea of the proof: The general idea is if the anti-canonical degree is large, then the curve C is deformable i.e. $\dim_C \mathrm{Chow}(X) > 0$.

We then prove that the deformation of the curve contains in a component S_i of negative par of Zariski decomposition $N(K_X)$. (Note that the surface S_i has K_{S_i} not pseudo-effective, thus it is a uniruled surface)

Therefore, we reduce the problem onto the surface S_i. We try to prove that K_{S_i}-negative curve on the uniruled surface breaks and produce a rational curve.

Proof. □

3 Cao-Höring's approach produce rational curve for Kähler manifold

3.1 Pseudo effectiveness of the relative adjoint class

The major technical tools that will be used in the proof of the Cao-Höring's theorem is the following pseudo-effectiveness theorem.

Theorem 3.

Remark 4.

3.2 Cao-Höring's main theorem

Now we can prove the main theorem of the [CH20].

References

[CH20] Junyan Cao and Andreas Höring, *Rational curves on compact Kähler manifolds*, J. Differential Geom. **114** (2020), no. 1, 1–39.

[HP16] Andreas Höring and Thomas Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.