Uniruled Kähler Manifolds and the MRC Fibration

Kähler MMP

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Big picture

Uniruledness detects the negative side of the canonical class. The MRC fibration separates the rationally connected directions from the non-uniruled base. Its importance for the Kähler MMP is practical: after choosing a suitable model, this fibration is projective, so the relative tools developed for analytic projective morphisms become available even when the total space is not projective.

1 The canonical class detects covering rational curves

Theorem 1.1 (BDPP; Ou in the compact Kähler case)**.** *A smooth compact Kähler manifold X is uniruled if and only if K_X is not pseudoeffective. See [BDPP13, Ou25] and the rational curves chapter for the proof architecture.*

The nontrivial direction produces covering rational curves from failure of positivity. It is not enough to find one curve of negative canonical degree: exceptional rational curves on a non-uniruled variety already show the distinction.

One must also distinguish K_X from an adjoint divisor. On $\mathbb{P}^1$, a boundary consisting of four distinct points each with coefficient $1/2$ is klt and makes $K_{\mathbb{P}^1} + B$ have degree zero. Thus an uniruled space can carry a pseudoeffective, even trivial, adjoint divisor. The MMP for pairs follows $K_X + B$.

2 The MRC quotient and the projective bridge

The maximal rationally connected quotient $X \dashrightarrow Z$ has rationally connected general fibres, and a rational curve through a very general point of X stays in its fibre. This maximality is the property used below. Resolve the graph and the base to obtain a smooth holomorphic Kähler model $f : X' \to Z'$.

Theorem 2.1 (Claudon–Höring)**.** *Every smooth holomorphic model of the MRC fibration of a compact Kähler manifold is a projective morphism. More generally a compact Kähler space with klt singularities has a projective model of its MRC fibration [CH24].*

The underlying projectivity criterion says that a fibration between compact Kähler manifolds is projective when pullback induces an isomorphism on holomorphic two-forms. For the MRC fibration the rationally connected fibres have no such forms; the descent theorem for forms gives the required isomorphism. Both descent and the projectivity criterion are geometric inputs, not consequences of projectivity of each individual fibre.

Proposition 2.2. *The smooth base Z' of such a model is not uniruled. Consequently $K_{Z'}$ is pseudoeffective.*

Proof. Suppose instead that rational curves cover Z'. Choose one through a very general point and meeting the locus of rationally connected fibres. Normalize it, and form the main component of the pullback of X' which dominates this curve. Resolve this component. Since f is projective, the resulting smooth total space is projective over $\mathbb{P}^1$, hence projective. Its general fibre is rationally connected. The theorem of Graber–Harris–Starr, together with the usual rational-connectedness consequence, makes this total space rationally connected: its base and general fibre are rationally connected [GHS03].
Two very general points lying over different points of this rational curve can therefore be joined by a rational curve (or a rational chain suffices). Its image in X' connects different general MRC fibres, contradicting maximality. Thus Z' is not uniruled, and Theorem 1.1 gives the last assertion. ∎

3 A concrete model: a ruled space over a non-uniruled base

Let Z be a smooth compact Kähler manifold containing no rational curves, for example a complex torus, and put $X = \mathbb{P}^1 \times Z$. Any map $\mathbb{P}^1 \to X$ is constant after projection to Z. Each fibre of $X \to Z$ is rationally connected, so this projection is exactly the MRC quotient. It is projective even when Z, and hence X, is nonprojective.

For $F = \mathbb{P}^1 \times \{z\}$ one has $K_X \cdot F = -2$. If ω is a Kähler class, the equation $(K_X + t\omega) \cdot F = 0$ forces $t = 2/(\omega \cdot F)$. This illustrates how an adjoint class can become trivial in the rationally

connected direction. It does not prove that the class is globally nef or descends: those require the contraction and descent theorems in the base-point-free chapter.

4 Role in the MMP and research directions

The MRC base and a Mori-fibre-space base need not coincide. The former records all rational connectivity, while the latter is an elementary output of a chosen adjoint MMP and may itself be uniruled. The MRC construction is useful before running a relative program; it is not a substitute for that program.

For the transcendental base-point-free problem, the important questions are whether the adjoint class is trivial on the appropriate fibres, whether it descends as a Bott–Chern class, and whether the descended class is positive enough for induction. Projectivity of the MRC model addresses the available relative tools. It does not settle those three assertions by itself.

References

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