Transcendental Volume Spring 2025

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The aim of this note is to introduce the transcendental volume.

Contents

[1] **WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

1 Absolute continuous part, non-pluripolar product, mobile intersection number

2 Definition of volumes

Definition 1 (Volume of a line bundle, [Laz04, Definition 2.2.31])**.** Let X be an irreducible projective variety of dimension n, and let L be a line bundle on X. The volume of L is defined to be the non-negative real number

$$\mathrm{vol}(L) = \mathrm{vol}_X(L) = \limsup_{m\to\infty} \frac{h^0\left(X, L^{\otimes m}\right)}{m^n/n!}.$$

The volume $\mathrm{vol}(D) = \mathrm{vol}_X(D)$ of a Cartier divisor D is defined similarly, or by passing to $\mathcal{O}_X(D)$.

Definition 2 (Volume of a $\mathbb{R}$-divisor, [FKL16, p. 8])**.** If X is a normal projective variety, D is an $\mathbb{R}$-divisor and $n = \dim X$, then we define the volume of D by

$$\mathrm{vol}(X, D) = \limsup \frac{n! h^0(X, mD)}{m^n}$$

here $h^0(X, mD) = h^0(X, \lfloor mD \rfloor)$.

Remark 3. As mentioned in [FKL16],

Definition 4 (Volume of cohomology classes, [Bou02])**.** Let X be a compact Kähler n-fold. We define the volume of a cohomology class $\alpha \in H^{1,1}(X, \mathbf{R})$ by

$$v(\alpha) := \sup_{T\in\alpha} \int_X T_{ac}^n = \sup_{T\in\alpha} \int_{X\setminus \mathrm{Sing}(T)} T^n$$

for T ranging over the closed positive (1,1)-currents in α, in case α is pseudo=effective. If it is not, we set $v(\alpha) = 0$.

Remark 5 (Absolute continuous part)**.** The reason to take the absolute continuous part is because that n-fold product of arbitrary closed positive current T is not well defined (require locally bounded potential condition). Once we assume T absolute continuous, the product T_{ac}^n is well defined.

By [Car25, Proposition 3.4], when T is a current with analytic singularity, then the wedge product coinside with the non-pluriproduct

$$\langle T^n \rangle = T_{ac}^n.$$

Remark 6 (Volume on normal varieties)**.** On compact normal Kähler variety, we can first take a resolution $\pi : X' \to X$, and define the volume of a class $\alpha \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ to be

$$\mathrm{vol}(\alpha) = \sup_{T\in\pi^*(\alpha)} \int_{X'} T_{ac}^n$$

It's well defined, since volume function for a smooth variety is birational invariant (see Theorem 14).

Alternatively we have the following definition of volume.

Definition 7. Let $\mathcal{E}$ be the pseudo-effective cone of the Kähler variety X. The volume of a class $\alpha \in \mathcal{E}$, denoted $\operatorname{vol}(\alpha)$, is defined as the supremum all numbers $\left(\tilde{\beta}^n\right)$ where $\mu : \tilde{X} \to X$ is a modification and $\tilde{\beta}$ is a Kähler class on $\tilde{X}$ such that $\tilde{\beta} \leq \mu^*(\alpha)$ (i.e., $\mu^*(\alpha) - \tilde{\beta}$ is pseudo-effective). When α is not pseudo-effective we define its volume to be zero.

3 Fujita's approximation

One of the central property that volume function satisfies is the Fujita approximation. Both big line bundles and big cohomology classes satisfy such property.

Theorem 8 (Fujita approximation (for big line bundle on projective variety))**.** Let L be a big line bundle on a projective manifold X. Then, for every $\varepsilon > 0$, there exists a modification $\mu : \tilde{X} \to X$, an ample $\mathbf{Q}$-line bundle A and an effective $\mathbf{Q}$-divisor E on $\widetilde{X}$ (the data depends on ε) such that:
(i) $L = A + E$ as $\mathbf{Q}$-line bundles,
(ii) $|v(A) - v(L)| < \varepsilon$, in particular

$$A^n > v(L) - \epsilon$$

Theorem 9 (Fujita approximation (for big cohomology classes on compact Kähler manifold), [Bou02, Theorem 1.4])**.** Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X, \mathbf{R})$ be a big class on X. Then, for every $\varepsilon > 0$, there exists a modification $\mu : \tilde{X} \to X$, a Kähler class ω and an effective real divisor D on $\tilde{X}$ such that
(i) $\mu^\star \alpha = \omega + \{D\}$ as cohomology classes,
(ii) $|v(\alpha) - v(\omega)| < \varepsilon$.

4 Basic properties of volume functions

4.1 For ($\mathbb{R}$-)divisors

Volume depends only on the numerical class of a $\mathbb{R}$-divisor.

Proposition 10 (Volume is numerical invariant, [FKL16, Theorem 3.5])**.** Let D be an $\mathbb{R}$-divisor on a proper normal variety X of dimension n. If D' is an $\mathbb{R}$-divisor on X such that $D' - D$ is a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then $\operatorname{vol}(D) = \operatorname{vol}(D')$.

Volume function is continnuous on the $N^1(X)_{\mathbb{R}}$ space.

Proposition 11 (Continuity of volume function, [Laz04, Corollary 2.2.45])**.** The function $\xi \mapsto \operatorname{vol}(\xi)$ on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a homogenuous continuous function

$$\operatorname{vol} : N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

Volume increase in effective directions.

Proposition 12 (Volume increases in effective directions, [Laz04, Example 2.2.48])**.** If $\xi \in N^1(X)_{\mathbf{R}}$ is big and $e \in N^1(X)_{\mathbf{R}}$ is effective, then

$$\operatorname{vol}(\xi) \leq \operatorname{vol}(\xi + e).$$

Remark 13 (Volume in pseudo-effective direction)**.**

Volume satisfies the birational invariance property.

Theorem 14. Let

$$\nu : X' \longrightarrow X$$

be a birational projective mapping of irreducible varieties. Then

$$\operatorname{vol}_X(\xi) = \operatorname{vol}_{X'}\left(\nu^* \xi\right)$$

for any class $\xi \in N^1(X)_{\mathbf{R}}$.

For generic finite morphism we have similar pull back formula.

Theorem 15. Let $f : Y \to X$ be a proper, dominant, generically finite morphism of normal projective varieties over k. For any $D \in \operatorname{Div}(X)$, we have

$$\operatorname{vol}_Y\left(f^* D\right) = \operatorname{deg}(f) \operatorname{vol}_X(D).$$

Similar to the global section, volume function satisfies the following property.

Theorem 16. Let $g : X \to Y$ be a birational morphism of normal projective varieties, and let D be a $\mathbb{R}$-Cartier divisor on Y and G be a $\mathbb{R}$-divisor on X. If

$$G - g^* D \geq 0$$

is effective and g-exceptional, then

$$\operatorname{vol}(Y, G) = \operatorname{vol}(X, D).$$

4.2 For cohomology classes

Volume function is continuous on the Bott-Chern cohomology space.

Proposition 17 ([Bou02, Corollary 4.11])**.** The volume $\operatorname{vol} : H^{1,1}(X, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

The volume function is log concave function.

Proposition 18 (log concavity of volume function, proved by Hacon)**.** Let X be a compact Kähler manifold. If $\alpha, \alpha' \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ are big classes, then the volume satisfies the log concavity property

$$\operatorname{vol}(\alpha + \alpha')^{1/n} \geq \operatorname{vol}(\alpha)^{1/n} + \operatorname{vol}(\alpha')^{1/n}.$$

Proof. □

5 Positivity, singularity and volume

Volume of a nef class can be computed using

6 Upper semi-continuity of volumes

Jiao proved the following upper semi-continuity of volume for projective family with irreducible reduced fibers.

Theorem 19 ([Jia25])**.**

Boucksom proved the following upper semi-continuity of volume for smooth Kähler family.

Theorem 20 ([Bou02])**.**

7 Deformation invariance of volume of adjoint divisors (classes)

Volume of log canonical divisor satisfies deformation invariance properties.

Theorem 21 ([HMX18, Corollary 4.3])**.** Let $\pi : X \longrightarrow T$ be a projective morphism of smooth varieties. Suppose that (X, Δ) is log canonical and has simple normal crossings over T. Then the volume function

$$t \longmapsto \mathrm{vol}(X_t, K_{X_t} + \Delta_t)$$

is independent of t.

This is also true for generalized Kähler pairs, when central fiber is projective with big adjoint class.

Theorem 22.

8 Volume in divisorial Zariski decomposition

We already seen that volume may increase in

9 Volume in the minimal model program

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