Moishezon space and Moizhezon morphism Summer 2025

Note 1 — 2025-07-07 (draft version)

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1 Overview

The aim of this note is to give a brief introduction to Moishezon vairety and Moishezon morphism. The major references for this note are [Kol22], [Fuj83], and [Uen75].

Why study Moishezon morphisms? First, Moishezon spaces have more functorial behavior (compared with projective varieties), as we will see in Section 2. Secondly, from almost any projective variety we can construct a Moishezon space via bimeromorphic modification, making Moishezon spaces versatile in birational geometry. Thirdly, by Artin's fundamental theorem, the category of Moishezon spaces appears naturally in moduli theory. Another compelling reason to consider the Moishezon category is that it allows cut-and-paste operations similar to those we can perform in topology.

This series of talks is organized as follows:

Lec 1. Basic knowledge about Moishezon spaces and Moishezon morphisms,

Lec 2. Fiberwise bimeromorphic problems.

Lec 3. General type locus, Moishezon locus, and projective locus.

Lec 4. Projectivity critera and behavior of projective locus.

Lec 5. Rational curves on Moishezon spaces.

Contents

2 Moishezon spaces

Definition 2.1 (Meromorphic S-map)**.** Let X, Y be reduced complex spaces. We call the S-map a *meromorphic S-map* if

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

the natural projection associated to the graph $p : \Gamma \to X$ is a proper bimeromorphic morphism. Moreover, if the natural projection $q : \Gamma \to Y$ is also a proper bimeromorphic morphism, then we call α a *proper bimeromorphic S-map.*

Remark 2.2 (Comparison between meromorphic map and S-meromorphic map)**.** By definition,

$$X \times_S Y \hookrightarrow X \times Y$$

is an inclusion. Therefore, it is easy to see that

$$S\text{-meromorphic map} \implies \text{meromorphic map.}$$

Conversely, the graph of a meromorphic map $\Gamma \subset X \times Y$ needs not to contain in $X \times_S Y$, so that a meromorphic map needs not to be a S-meromorphic map.

Remark 2.3 (Comaprison between S-meromorphic map and fiberwise meromorphic map)**.** Note that a S-bimeromorphic map does not need to be a fiberwise bimeromorphic map. Since the restriction of a bimeromorphic map on the subvariety (the fiber) need not to be a bimeromorphic map. We will discuss more about the fiberwise bimeromorphic map in the Note-2.

Definition 2.4 (Moishezon space, first definition)**.** A proper, irreducible, reduced analytic space X is Moishezon if it is bimeromorphic to a projective variety X^{p}.

Remark 2.5. The following proposition tell us when the meromorphic map is an actual morphism, using the rigidty lemma.

Let $f : X \dashrightarrow Y$ be a bimeromorphic map with the resolution of indetermancy.

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

then if any $C \subset W$ p-exceptional is also q-exceptional. Then the birational map is also a morphism.

Definition 2.6 (Moishezon space, second definition)**.** A proper, irreducible, reduced analytic space X is Moishezon if

$$a(X) := \operatorname{tr}\deg_{\mathbb{C}} M(X) = \dim(X)$$

that is, it has $\dim X$ number of algebraic dependent meromorphic function.

Definition 2.7 (Moishezon space, third definition)**.** A proper irreducible, reduced analytic space X is Moishezon if it carries a big rank 1 reflexive sheaf $\mathscr{F}$. Here the big rank 1 reflexive sheaf means that the induced Kodaira map $g : X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{F}))$ is bimeromorphic onto it's image.

Proposition 2.8. Three different definitions for Moishezon spaces above are equivalent.

Proof. see e.g. [Uen75]. ∎

The first important property for Moishezon space is that it locally looks like quasi-projective scheme up to a étale cover.

Proposition 2.9 ([Kol22, Proposition 8.2])**.** Let X be a Moishezon space. For every $x \in X$ there is a pointed quasi-projective scheme (x', X') and an étale morphism $(x', X') \to (x, X)$.

Proof. It's quite difficult; for the sake of time, we omit it here. For the curious reader, please refer to [Art70]. ∎

Lemma 2.10 (Existence of Galois closure)**.** Let $\pi : X' \to X$ be a finite covering between normal analytic varieties. Then there exists a finite Galois covering $\varphi : X'' \to X$ from a normal analytic variety X'' which factors through π which is universal in the following sense:

$$\begin{tikzcd} & X' \arrow[dr, "\pi"] & \\ X'' \arrow[ur] \arrow[rr, "\varphi"'] & & X \end{tikzcd}$$

For any finite Galois covering $\psi : Y \to X$ from a normal analytic variety which factors through π, there exists uniquely a Galois covering $Y \to X''$ over X'.

Using the existence of Galois closure, we can write a normal Moishezon space globally as a quotient of a proper variety by a finite group.

Proposition 2.11 ([Kol22, Proposition 8.3])**.** Let X be a Mosiehzon variety. If X is normal, then there is a proper variety Y and a finite group G that acts on Y such that $X \cong Y/G$. (Note that in general Y can not be chosen projective.)

Proof. First, by Proposition 2.11, there exists some étale cover of X (indeed, since the étale morphism is finite, we can find an open cover of X be the étale morphism). Since X is proper, we can find some finite cover of it. Now by the previous lemma we can take the Galois closure of the finite étale cover $X_i \to X$. We then apply the universal property of the Galois closure, thus it is possible to patch the collection of Galois closures $\{X_i \to X\}$ together in the Zariski topology via gluing lemma (see e.g. Hartshorne Exercise II 2.12.), and therefore we can get a finite covering of X, $Y \to X$ and thus $X \simeq Y/G$.

∎

Artin [Art70] proved the following theorem, demonstrating the importance of the category of Moishezon spaces in moduli theory.

Proposition 2.12 ([Art70, Theorem 7.3])**.** There is a natural functor

$$\text{an} : (\text{algebraic space of finite type over } \mathbb{C}) \to (\text{complex spaces})$$

extending the functor an on the category (schemes of finite type /$\mathbb{C}$). This functor induces an equivalence of categories

$$\text{(complex algebraic schemes of finite type}/\mathbb{C}) \to \text{(Moishezon spaces)}.$$

In other words, every Moishezon space is in an unique way an algebraic space.

We next prove that a Kahler Moishezon space with 1-rational singularity is a projective variety. Before proving the theorem, let us first state two results that will be used in the proof.

Lemma 2.13. Let X be a compact Moishezon space with 1-rational singularity, that is, X is normal and has a resolution $\pi : Y \to X$ such that $R^1\pi_*\mathcal{O}_Y = 0$. Then an analytic homology class $b \in A_2(X, \mathbb{Q})$ is zero if it is numerically equivalent to 0. In particular,

$$A_2(X, \mathbb{Q}) = N_1(X)_{\mathbb{Q}} \subset H^2(X, \mathbb{Q}).$$

Lemma 2.14 (Nakai-Moishezon criterion for $\mathbb{Q}$-line bundles over Kähler Moishezon space)**.** Let X be a Kähler Moishezon space with a Kähler form ω. Assume that an element $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ satisfies the equality for any curve $C \subset X$:

$$(C.L) = \int_C \omega.$$

Then L is ample.

Proposition 2.15 ([Nam02])**.** Let X be a Moishezon space with 1-rational singularity. If X is Kähler, then X is projective.

Proof of the Proposition 2.15. Since the numerical equivalence and the homological equivalence coincide for (analytic) 1-cycle by Lemma 2.13, we have a natural map

$$\alpha : N^1(X)_{\mathbb{Q}} \to (A_2(X, \mathbb{Q}))^*, \quad d \mapsto (- \cdot d),$$

and α is an isomorphism (by duality of $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{Q}}$).

Note that $\omega \in H^2(X, \mathbb{R})$ Kähler form as an element of $(A_2(X, \mathbb{R}))^*$. By simply define

$$\alpha_\omega : A_2(X, \mathbb{R}) \to \mathbb{R}, \quad C \mapsto \omega \cdot C = \int_C \omega.$$

Since $\alpha_{\mathbb{R}}$ is surjective, there is an element $d \in N^1(X)_{\mathbb{R}}$ such that

$$(C \cdot d) = \int_C \omega,$$

for every curve $C \subset X$.

We then approximate $d \in N^1(X)_{\mathbb{R}}$ by a convergent sequence $\{d_m\}$ of rational elements $d_m \in N^1(X)_{\mathbb{Q}}$.

Let us fix the basis $b_1, \ldots, b_l$ of the vector space $N^1(X)_{\mathbb{Q}}$. Each b_i is represented by an element $B_i \in \mathrm{Pic}(X)_{\mathbb{Q}}$ via the quotient

$$\mathrm{Pic}(X)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}} = \mathrm{Pic}(X)_{\mathbb{Q}}/\equiv, \quad B_i \mapsto b_i,$$

Now d (resp. d_m) is represented by an element in $\mathrm{Pic}(X)_{\mathbb{R}}$ (resp. $\mathrm{Pic}(X)_{\mathbb{Q}}$)

$$D := \Sigma x_i B_i,$$

(resp. $D_m := \Sigma x_i^{(m)} B_i$) such that $\lim x_i^{(m)} = x_i$. Put $E_m := D_m - D$. Then there are d closed $(1,1)$-forms α_m corresponding to E_m such that $\{\alpha_m\}$ uniformly converge to 0 .

If m is chosen sufficiently large, then $\omega_m := \omega + \alpha_m$ is a Kähler form. Since

$$(C.D_m) = \int_C \omega_m > 0,$$

for every curve $C \subset X$. We see that D_m is ample by Lemma 2.14 (Note that we have D_m being a $\mathbb{Q}$-divisor, so that it's possible to apply the Nakai-Moishezon criterion). In particular, X is projective. □

Remark 2.16. There exist some Kähler Moishezon spaces with bad singularity that are not projective. (As we shall see in the last section).

Proposition 2.17 ([Kol22, Proposition 8])**.**
(1) Let X be a Moishezon space, if $Z \to X$ be finite then Z is Moishezon.

(2) Let X be a Moishezon space, and $f : X \to Y$ be a surjective morphism of complex varieties. Then Y is also Moishezon.

(3) Let X be a Moishezon space, assume that $Z \subset X$ is Mosiezhon, then

Proof of (1). By definition

$$\mathrm{trdeg}_{\mathbb{C}} K(X) = \dim X,$$

and if Z is finite map then

$$K(X) \hookrightarrow K(Z),$$

is a finite field extension. Therefore by additive property for a tower of field extensions, we have

$$\mathrm{trdeg}_{\mathbb{C}}(K(Z)) = \mathrm{trdeg}_{\mathbb{C}}(K(X)) + \mathrm{trdeg}_{K(X)} K(Z) = \mathrm{trdeg}_{\mathbb{C}}(K(X)).$$

□

Proof of (2). It will be generalized in to the relative version, see 3.14. □

Proof of (3). Consider the following pull back diagram.

$$\begin{tikzcd} Z^p = f^{-1}(Z) \arrow[r] \arrow[d, "f_Z"'] & X^p \arrow[d, "f"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

Clearly Z^p is projective (as subvariety of X^p), and f_Z is surjective (by definition of Z^p). Therefore, by (2), we know that Z is again Moishezon. □

The following proposition shows that the Moishezon manifolds admit strong Hodge decomposition.

Proposition 2.18. If X is a Moishezon manifold, then the Hodge decomposition holds, indeed a Moishezon manifold admits strong Hodge decomposition.

Before proving the theorem, let us first define what is strong Hodge decomposition. We say that a compact manifold admits a *strong Hodge decomposition* if the natural maps

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\ [\alpha^{p,q}]_{\mathrm{BC}} \mapsto [\alpha^{p,q}]_{\bar\partial} \quad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C}), \quad \sum[\alpha^{p,q}] \mapsto \sum \alpha^{p,q},$$

are isomorphisms.

Remark 2.19. As a direct consequence, we see that a Moishezon manifolds admits *the Du Bois property*, that is

$$H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X),$$

is surjective for all $i \geq 0$. (which will be used in the third note).

Proof. The idea of the proof comes from [Dem97, Proposition (12.3)]. We first take the projective modification

$$\mu : \widetilde{X} \to X,$$

such that X' is a projective manifold. And therefore X' admits a strong Hodge decomposition. On the other hand

We first observe that $\mu_\star\mu^\star\beta = \beta$ for every smooth form β on Y. In fact, this property is equivalent to the equality

$$\int_Y (\mu_\star\mu^\star\beta)\wedge\alpha = \int_X \mu^\star(\beta\wedge\alpha) = \int_Y \beta\wedge\alpha.$$

for every smooth form α on Y, and this equality is clear because μ is a biholomorphism outside sets of Lebesgue measure 0 (which holds in general for a proper surjective bimeromorphic map).

Consequently, the induced cohomology morphism $\mu_\star$ is surjective and $\mu^\star$ is injective (but these maps need not be isomorphisms).

$$\begin{array}{cccc}
H^{p,q}_{\mathrm{BC}}(\widetilde{X},\mathbb{C}) \longrightarrow H^{p,q}(\widetilde{X},\mathbb{C}), & & \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(\widetilde{X},\mathbb{C}) \longrightarrow H^k(\widetilde{X},\mathbb{C}) \\
\mu_\star\downarrow\uparrow\mu^\star \qquad \mu_\star\downarrow\uparrow\mu^\star & & \mu_\star\downarrow\uparrow\mu^\star \qquad \mu_\star\downarrow\uparrow\mu^\star \\
H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}), & & \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C})
\end{array}$$

Now, we have commutative diagrams with either upward or downward vertical arrows. Hence the surjectivity or injectivity of the top horizontal arrows implies that of the bottom horizontal arrows.

□

We next introduce Campana's Moishezon criterion. The proof uses the core reduction he introduced.

Proposition 2.20 ([Cam81, Corollaire on p. 212])**.** Let X be a compact complex variety in the Fujiki class $\mathscr{C}$. Then X is Moishezon if and only if X is algebraically connected.

As an immediate consequence.

Corollary 2.21. A compact Kahler manifold is projective iff it's algebraically connected.

Proposition 2.22. Let $f : X \to B$ be a fibration over an algebraically connected variety (e.g. a projective curve). Assume that X is in the Fujiki class $\mathscr{C}$ and the general fiber of f is algebraically connected, then X is Moishezon if and only if f has a multi-section.

Proof. The proof is clear, since admit multi-section implies the algebraic connectedness of X. □

Remark 2.23. For readers interested in the applications of the algebraic connectedness criterion, I recommend the paper by [Lin23]. He try to addresse the following question.

Question 2.24 (Oguiso–Peternell problem, [Lin23, Problem 1.2])**.** Let X be a compact Kähler manifold of dimension n such that $\operatorname{Int}(\operatorname{Psef}(X)^\vee)$ (or $\operatorname{Int}(\mathscr{K}(X)^\vee)$ for dual Kähler cone $\mathscr{K}(X)$) contains an element of $H^{2n-2}(X, \mathbb{Q})$. Is X always projective? If not, how algebraic is X ?

3 Moishezon morphisms

Let us first recall the definition of a projective morphism.

Definition 3.1 (Projective morphism, first definition)**.** Let $X \to S$ be a proper morphism between complex spaces. f is projective if there exists a locally free coherent sheaf $\mathcal{E}$ of finite rank such that there exists a closed S-immersion $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$, with the following diagram commute.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}_S(\mathcal{E}) \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.2 (Projective morphism, second definition)**.** Let $X \to S$ be a proper morphism between complex spaces. f is projective if X can be embedded in $\mathbb{P}^N \times S$ for some N, with the following the diagram commute.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}^N \times S \arrow[dl] \\ & S & \end{tikzcd}$$

Note that Kollár adopt the second definition.

Definition 3.3 (Locally projective morphism)**.** Let $f : X \to S$ be a proper morphism of complex spaces. We call f locally projective if for every relatively compact open subset Q of S the restriction $f_Q : X_Q \to Q$ is a projective morphism.

Remark 3.4. Easy to see the second definition will immediate implies the first definition. Converse direction also holds when the base is Stein or quasi-projective.

Proof. Assume we have the 1st definition, so that $f : X \to S$ and $g : Y = \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \to S$. Let A be an g-ample line bundle. And, therefore by Serre vanishing theorem over some Stein compact subset $B \subset S$, for some sufficient large $n \gg 0$, we have

$$g^* g_*(\mathcal{E} \otimes A^{\otimes n}) \to \mathcal{E} \otimes A^{\otimes n},$$

is surjective. Since the base S is Stein, by Cartan A theorem, $g_*(\mathcal{E} \otimes A^{\otimes n})$ is global generated. And therefore so it's the pull back $g^* g_*(\mathcal{E} \otimes A^{\otimes n})$. Since the surjective map sends global generated coherent sheaf to global generated coherent sheaf. This means that $\mathcal{E} \otimes A^{\otimes n}$ is global generated.

By coherence of $\mathcal{E} \otimes A^{\otimes n}$, the cohomology group $V = H^0(Y, \mathcal{E} \otimes A^{\otimes n})$ is finite dimensional. And there is a surjection

$$V \otimes \mathcal{O}_Y \to \mathcal{E} \otimes A^{\otimes n}.$$

And therefore it will induce an embedding

$$X \hookrightarrow \mathbb{P}_B(\mathcal{E}) = \mathbb{P}_B\left(\mathcal{E} \otimes A^{\otimes m}\right) \hookrightarrow \mathbb{P}(V) \times B,$$

after shrink the base $B \subset S$. □

Remark 3.5. When the total space has only finite number of irreducible components, then a locally projective morphism is bimeromorphic to a projective morphism. (see [Fuj83, Lemma 1.3.1]).

In what follows, we may assume that the base S is reduced. However, in general, we do not require the total space X to be reduced or not.

Definition 3.6 (Moishezon morphism, 1st definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if $g : X \to S$ is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

That is, there is a closed subspace $Y \subset X \times_S X^{\mathrm{p}}$ such that the coordinate projections $Y \to X$ and $Y \to X^{\mathrm{p}}$ are bimeromorphic.

$$\begin{tikzcd}
& Y \arrow[dl] \arrow[dr] & \\
X \arrow[rr, dashed] \arrow[dr] & & X^p \arrow[dl] \\
& S &
\end{tikzcd}$$

Definition 3.7 (Moishezon morphism, 2nd definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if There is a projective morphism of algebraic varieties $G : \mathbb{X} \to \mathbb{S}$ and a meromorphic $\phi_S : S \dashrightarrow \mathbb{S}$ such that X is bimeromorphic to $\mathbb{X} \times_{\mathbb{S}} S$, the fiber product of rational maps is defined where the maps are defined, so on a dense open set.

Remark 3.8. Let us say few words about the fiber product for a rational map $\phi_S : S \dashrightarrow \mathbb{S}$, the fiber product is defined on the place that ϕ_S is holomorphic map.

Definition 3.9 (Moishezon morphism, 3rd definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if there is a rank 1, reflexive sheaf L on X such that the natural map $X \dashrightarrow \mathrm{Proj}_S\,(g_*L)$ is bimeromorphic onto the closure of its image.

Proposition 3.10. Three definitions of Moishezon morphism are equivalent.

Proof. Definition 3.7 equivalent to the Definition 3.6 is clear (using Proposition 3.16). Conversely, if there exists a projective family $X^p \to S$ that bimeromorphic to a given $f : X \to S$, then by generic flatness we know $g^p : X^p \to S$ is flat over S^o for some Zariski open subset $S^o \subset S$, and therefore using the definition of projective family, there exist a morphism

$$S^o \to \mathrm{Hilb}(\mathbb{P}^N)$$

such that the projective family is the pull back

$$\begin{tikzcd} X^p \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ S \arrow[r, dashed] & \mathrm{Hilb}(\mathbb{P}^N) \end{tikzcd}$$

We now show that the first definition and third definition are equivalent. From third definition to first definition is clear since $\mathrm{Proj}_S(f_*L)$ is projective over S. Conversely, if $f : X \to S$ is bimeromorphic to a projective morphism $X^p \to S$. Then since we assume X is normal, therefore the meromorphic map $X \dashrightarrow X^p$ is morphism outside codimension 2 subset. And the pull back $(\phi^o)^*\mathcal{O}_X(1)$ is a big line bundle defined on a big open subset, which can be extended uniquely to a big rank 1 reflexive sheaf. $\square$

Remark 3.11. The termiology in different paper are different, we can summarize it as below.

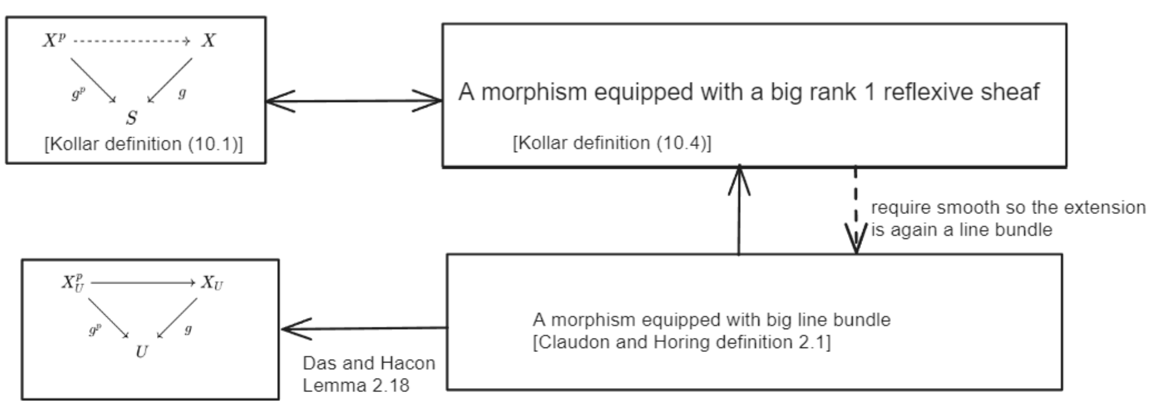

Figure 1: Definitions in different papers

Moishezon morphism satisfies the following Chow type lemma (which can be viewed as the deterministic property of a Moishezon morphism).

Theorem 3.12 ([DH20, Lemma 2.18])**.** Let $f : X \to S$ be a proper surjective morphism of analytic varieties, and let L be a f-big line bundle on X and D a $\mathbb{Q}$-divisor. Then

(1) Over any relatively compact open subset $V \subset S$, there exists a proper (indeed it's projective see [CH24]) bimeromorphic morphism $\alpha : W \to f^{-1}V$ from a smooth analytic variety W such that

$\beta = f|_{f^{-1}V} \circ \alpha : W \to V$ is a projective morphism and,
(2) $\left(W, \alpha_*^{-1}\left(D|_{f^{-1}V}\right) + \mathrm{Ex}(\alpha)\right)$ is a log smooth pair.

First, let us compare the theorem above with the Definition 3.6, in the Definition, we only assume the existence of some bimeromorphic S-map, the Chow lemma allows us to choose some bimeromorphic projective morphism.

Proof. Let $\phi : X \dashrightarrow Y$ be the relative Iitaka fibration of L over S and $g : Y \to S$ the induced projective morphism. Since L is f-big, $\phi : X \dashrightarrow Y$ is bimeromorphic. Let $p : \Gamma \to X$ and $q : \Gamma \to Y$ be the resolution of indeterminacy of ϕ so that p is proper.

$$\begin{tikzcd}
& \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

Now fix a relatively compact open subset $V \subset S$. Choose another relatively compact open set $U \subset S$ containing V such that $\bar{V} \subset U$. Note that U is σ-compact, since it is relatively compact. Since f and g are both proper morphisms, it follows that $X_U := f^{-1}U$ and $Y_U := g^{-1}U$ are both σ-compact. Let $\Gamma_U := q^{-1}\left(g^{-1}U\right) = p^{-1}\left(f^{-1}U\right)$. Then from the commutative diagram above it follows that $q|_{\Gamma_U} : \Gamma_U \to g^{-1}U$ is a proper morphism. In particular, Γ_U is σ-compact. Note that $q|_{\Gamma_U}$ is bimeromorphic. Therefore there is a projective bimeromorphic morphism $h : Z \to \Gamma_U$ from an analytic variety Z such that $q|_{\Gamma_U} \circ h : Z \to Y_U$ is a projective bimeromorphic morphism. Since g is projective, so is $Z \to U$.

Now we replace U by our previously fixed open set V. Then $Z_V := (g \circ\, q \circ h)^{-1}V$ is a relatively compact open subset of Z. Let $r : W \to Z_V$ be the log resolution of $\left(Z_V, (p \circ h)_*^{-1}\left(D|_{f^{-1}V}\right)\right)$. Let $\alpha := p|_{\Gamma_V} \circ h\Big|_{h^{-1}\Gamma_V} \circ r$ and $\beta := g|_{g^{-1}V} \circ q\Big|_{\Gamma_V} \circ h\Big|_{h^{-1}\Gamma_V} \circ r$, where $\Gamma_V := p^{-1}\left(f^{-1}V\right) = q^{-1}\left(g^{-1}V\right)$. Note that β is a projective morphism, since it is a componsition of projective morphisms over relatively compact bases.

Then $\alpha : W \to f^{-1}V$ is a proper bimeromorpic morphism and $\beta : W \to V$ is a projective morphism such that $\beta = f|_{f^{-1}V} \circ \alpha$ and $\left(W, \alpha_*^{-1}\left(D|_{f^{-1}V}\right) + \mathrm{Ex}(\alpha)\right)$ is a log smooth pair. $\square$

Proposition 3.13 ([Fuj83, Proposition 1.5.(4)])**.** Suppose that there exists a locally projective morphism $g : Y \to S$ and a generically finite meromorphic S-map $h : X \dashrightarrow Y$. Then f is Moishezon.

$$\begin{tikzcd}
X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

Proof. First since being Moishezon is stable under bimeromorphic change, without lose of generality we can assume that h is a morphism. Since Moishezon morphism and locally projective morphism are proper. So that h is proper. Apply the Stein factorization theorem, such that h_2 is projective (since h_2 is finite) and h_1 is proper. Thus, the composition $g \circ h_2$ is locally projective. And thus by definition $X \to S$ is a Moishezon morphism.

$$\begin{tikzcd} & X^* \arrow[dr, "h_2"] & \\ X \arrow[ur, "h_1"] \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

□

Proposition 3.14 ([Fuj83, Proposition 1.7]). Let $f : X \to S$ be a Moishezon morphism, and $g : Y \to S$ a proper morphism, of reduced complex spaces. Suppose that there is a generically surjective meromorphic S-map $h : X \dashrightarrow Y$. Then g also is Moishezon.

Proof. This Proposition can be viewed as a generalization of the Proposition 2.17. The proof is a bit involving, and we omit it here. □

Proposition 3.15 ([Fuj83, Proposition 1.5]).
(1) The morphism $f : X \to S$ is Moishezon if and only if for each irreducible component X_i of X the restriction $f = f|_{X_i} : X_i \to S$ is Moishezon.

(2) Let $f : X \to S$ be a Moishezon morphism. Then: For every reduced analytic subspace $X' \subseteq X$ the induced morphism $f' = f|_{X'} : X' \to S$ is Moishezon.

Proof. For (1), let's take the normalization

$$\nu : X^\nu \to X,$$

recall that for a reduced complex space with finite many irreducible component, the normalization is a bimeromorphic map. So that $f : X \to S$ is Moishezon iff the restriction on each component X_i are Moishezon.

For (2), by the Chow lemma (Theorem 3.12), we can find some locally projective morphism such that X^* is smooth and h is a bimeromorphic S-morphism.

$$\begin{tikzcd} X^* \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

We then take the inverse image of the analytic subspace X' denote it $Z = h^{-1}(X')$. (we can assume the inverse image has reduced structure). Since the restriction of the projective morphism on $g|_Z : Z \to S$ is still locally projective. And by construction, clearly the morphism $Z \to X'$ is surjective. And therefore, by Proposition 3.14, we know that $X' \to S$ is a Moishezon morphism. □

Restriction on the image side will also preserve the Moishezon condition.

Proposition 3.16 (A morphism is Moishezon iff it's Moisheozn onto its image)**.** Let $f : X \to S$ be a proper morphism between analytic spaces, let $f' : X \to f(X) = Y \subset S$ be the restriction, then f is Moishezon (resp. projective) iff f' is Moishezon (resp. projective).

Proof. It's enough to prove the case for projective morphism case (and Moishezon morphism case follows easily).

To see this, assume that we the morphism $f : X \to S$ is projective, by definition there it factor through the $X \hookrightarrow \mathbb{P}^n_S \to S$. Doing base change on $Y \hookrightarrow S$, proves the projective of $X \hookrightarrow \mathbb{P}_Y \to Y$.

Converse direction is clear, since The composition $X \to f(X) \hookrightarrow S$ can be written as:

$$X \hookrightarrow \mathbb{P}^n_{f(X)} \to f(X) \hookrightarrow S.$$

Since $\mathbb{P}^n_{f(X)} = \mathbb{P}^n_S \times_S f(X)$, we can rewrite the morphism as:

$$X \hookrightarrow \mathbb{P}^n_S \times_S f(X) \hookrightarrow \mathbb{P}^n_S \to S,$$

where the second second inclusion is because $f(X) \hookrightarrow S$ and so it's the projective bundle. $\square$

Proposition 3.17. When the base is Moishezon then the total space is Moishezon iff the morphism is Moishezon.

Proof. We first prove that morphism between Moishezon space is a Moishezon morpihsm. Let us define the graph embedding to be

$$\iota : X \to X \times S, \quad x \mapsto (x, f(x)),$$

since X is Moishezon it's bimeromorphic to a projective variety, as the diagram below shows

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & X \times S \arrow[r, dashed] \arrow[d, "\pi"] & X^p \times S \arrow[dl, "\pi^p"] \\ & S & \end{tikzcd}$$

Clearly, π^p is a projective morphism. And consequently π is a Moishezon morphism. And finally by Proposition 3.15, the morphism $f : X \to S$ is again Moishezon.

Conversely, if the morphism is Moishezon, and S is Moishezon space. Then there exist bimeromorphic modifications such that the following diagram commute

$$\begin{tikzcd} X^p \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[dl] \\ S^p \arrow[r] & S & \end{tikzcd}$$

Where $X' \to S$ is a projective morphism and S^p is a projective variety. Since the base change preserve the projective condition, easy to see that $X^p \to S^p$ is a projective morphism over S^p. And therefore X^p is a projective variety. By Proposition 3.14, X' is a Moishezon space. Since $X' \to X$ is bimeromorphic, this implies that X is also Moishezon. $\square$

Proposition 3.18 ([Kol22, Lemma 15])**.** Let $g : X \to S$ be a proper, generically finite, dominant morphism of normal, complex, analytic spaces. Then $\mathrm{Ex}(g) \to S$ is Moishezon.

Proof. We will prove the result under the additional assumption that S is Stein. By the geometric Noether normalization theorem, there exists a finite morphism

$$S \to \mathbb{C}^{\dim S}.$$

After replacing the base by $\mathbb{C}^{\dim S}$, we can assume that smooth locus of S is dense in $g(\mathrm{Ex}(g))$. Note that, by Proposition 3.13, if the restriction on $\mathbb{C}^{\dim S}$ is Moishezon, then so will the restriction on S. We will prove the result by induction on dimension.

We first define the base case $(g_0 : X_0 \to S_0) := (g : X \to S)$. Let E_0 be a g_0 exceptional divisor, with the image $Z_0 = g_0(E_0)$. We then inductively define the morphism $g_{i+1} : X_{i+1} \to S_{i+1}$ as follows. Assume that we already construct $g_i : X_i \to S_i$, we then blow up S_i along Z_i. We then blow up S_i along Z_i and let S_{i+1} be the normalization of the blow-up $\mathrm{Bl}_{Z_i} S_i$. Since S_i is reduced, this will induce a generic finite map $\phi : X_i \dashrightarrow S_{i+1}$. So that by the universal property of the normalization, the generic finite morphism $g_i : X_i \to S_i$ lift to a generic finite morphism $g_{i+1} : X_{i+1} \to S_{i+1}$, where X_{i+1} is the normalization of the graph of the map $\phi : X_i \dashrightarrow S_{i+1}$.

$$\begin{tikzcd}
X_{i+1} \arrow[r, "g_{i+1}"] \arrow[d] & S_{i+1} = \mathrm{Bl}_{Z_i}(S_i)^{\nu} \arrow[d] \\
\Gamma_\phi \arrow[ur] \arrow[d] & \mathrm{Bl}_{Z_i}(S_i) \arrow[d] \\
X_i \arrow[r, "g_i"'] \arrow[uur, dashed, "\phi"] & S_i
\end{tikzcd}$$

Let $E_{i+1} \subset X_{i+1}$ denote the bimeromorphic transform of E_i. (Note that $X_{i+1} \to X_i$ is an isomorphism over an open subset of E_i). We then compute the vanishing order $a(E_i, S_i)$ of Jacobian of g_i along E_i. We claim that

$$a\left(E_{i+1}, S_{i+1}\right) \leq a\left(E_i, S_i\right) + 1 - \operatorname{codim}\left(Z_i \subset S_i\right).$$

Thus eventually we reach the situation when $\operatorname{codim}(Z_i \subset S_i) = 1$, indeed if $\operatorname{codim}(Z_i \subset S_i) \geq 2$ then the Jacobian of g_i along E_i will eventually goes to zero. Contradiction.

Thus by comparing the dimension we know when restrict the morphism $X_i \to S_i$ to $E_i \to Z_i$ it will become a generic finite morphism. Since $S_{i+1} \to S_i$ is projective, the composition $Z_i \to Z_0$ will be a locally projective morphism.

$$\begin{tikzcd}
E_i \arrow[rr] \arrow[dr] & & Z_i \arrow[dl] \\
& Z_0 &
\end{tikzcd}$$

By Proposition 3.13, we know that $E_i \to Z_0$ is a Moishezon morphism. Since the strict transform $E_i \to E_0$ is a dominant morphism, by Proposition 3.14, we know that $E_0 \to Z_0$ is also Moishezon morphism. Finally, by Proposition 3.15 and Proposition 3.16, we know that $\mathrm{Ex}(f) \to S$ is Moishezon. □

Theorem 3.19 (Fibers of the Moishezon morphism are Moishezon spaces, [Kol22, Corollary 16])**.** The fibers of a proper, Moishezon morphism are Moishezon.

Proof. Let $g : X \to S$ be a proper, Moishezon morphism. It is bimeromorphic to a projective morphism $X^{\mathrm{p}} \to S$. We may assume X^{p} to be normal. Let Y be the normalization of the closure of the graph of $X \dashrightarrow X^{\mathrm{p}}$.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] \arrow[dr] & & X^p \arrow[dl] \\ & S & \end{tikzcd}$$

Fix now $s \in S$. Let $Z_s \subset X_s$ be an irreducible component, since given a proper dominant morphism, there exist at least one irreducible component dominant the base, there exist $W_s \subset Y_s$ an irreducible component that dominates Z_s. And by Proposition 3.14 and Proposition 3.15, it's enough to show that W_s is Moishezon. We divide the problem into two cases:

If $\pi : Y \to X^{\mathrm{p}}$ is generically an isomorphism along W_s, then W_s is bimeromorphic to an irreducible component of X_s^{p}, hence Moishezon.

Otherwise $W_s \subset \mathrm{Ex}(\pi)$. Now $\mathrm{Ex}(\pi) \to X^{\mathrm{p}}$ is Moishezon by Proposition 3.18. And by induction on dimension, since $\dim \mathrm{Ex}(\pi) < \dim X = \dim Y$, the fiber W_s is Moishezon. □

Proposition 3.20 ([Kol22, Example 13])**.** Let Z be a normal, projective variety with discrete automorphism group. Let $g : X \to S$ be a fiber bundle with fiber Z over a connected base S. Then g is Moishezon $\Leftrightarrow$ g is projective $\Leftrightarrow$ the monodromy is finite.

Remark 3.21. The monodromy here is different from the cohomological monodromy. Here the monodromy is refered as the fiber bundle monodromy

$$\rho : \pi_1(S) \to G \subset \mathrm{Aut}(Z)$$

where $G \subset \mathrm{Aut}(Z)$ is the structure group of the fiber (e.g. when the fiber bundle is principal G-bundle, then the structure group is simply the group G). Finite monodromy condition means that $\mathrm{im}(\rho) \subset G$ is a finite subgroup.

Before proving the theorem, let us state a lemma from fiber bundle theory, that is useful in what follows (which can be viewed as generalization of Ehersmann theorem over a simply connected base, when the base is simply connected, the fiber bundle is automattically trivial).

Lemma 3.22. Let $g : X \to S$ be a fiber bundle with trivial monodromy group, then the fiber bundle is actually a trivial fiber bundle.

The following proposition about the automorphism of polarized projective variety will be useful in the proof.

Proposition 3.23. Let Z be a projective variety, and L an ample line bundle over Z. Then $\mathrm{Aut}(Z, L) = \{\phi \in \mathrm{Aut}(Z) \mid c_1(\phi^* L) = c_1(L)\}$ is finite if the automorphism group $\mathrm{Aut}(Z)$ is discrete.

Proof. Let

$$\phi : Z \to Z \in \mathrm{Aut}(Z, L),$$

consider the graph $\Gamma_\phi \subset Z \times Z$, thus the Hilbert polynomial of Γ_ϕ relative to $L \boxtimes L$ is given by

$$H_\phi(n) = \chi(\Gamma_\phi, (L \boxtimes L)^n) = \chi(Z, L^{\otimes n} \otimes \phi^*(L^{\otimes n})).$$

On the other hand, since $c_1(\phi^* L) = c_1(L)$, so that easy to see that $L^{\otimes n} \otimes \phi^*(L^{\otimes -n})$ is numerical trivial, and thus

$$H(n) = \chi(Z, L^{\otimes n} \otimes \phi^*(L^{\otimes n}) \otimes L^{\otimes n} \otimes \phi^*(L^{\otimes -n})) = \chi(Z, L^{\otimes 2n}),$$

which is independent of ϕ and denote it $P(n) = H_\phi(n)$. So that the graph lies on $\mathrm{Hilb}^P_{Z\times Z}$ (with fixed Hilbert polynomial $P(n)$), which is of finite type. Thus contains finite many irreducible components (by Noetherian proprty). On the other hand, since Z is projective, each irreducible component of the Hilbert scheme is proper. Thus $\mathrm{Aut}(Z, L)$ is finite. □

The idea of the proof of the theorem is provided by Professor Kollár.

Proof of the theorem. Only needs to show that (1) implies (3) and (3) implies (2). For (3) implies (2), we try to take the étale base change so that the fiber bundle becomes trivial bundle. We can do as follows, Consider $\rho(\pi_1(S)) \subset \mathrm{Aut}(Z)$ is finite, let $S' \to S$ be the corresponding finite (unbranched) cover that kills the monodromy. Indeed since we have the

$$\rho : \pi_1(S) \to G$$

then the kernel of $\ker(\rho)$ is a subgroup of $\pi_1(S)$ is finite index, therefore by the Galois correspondence for covering, there exist finite étale cover of the base

$$\tilde{S} \to S,$$

such that monodromy of the fiber bundle under the base change becomes trivial, then by the previous lemma, after the base change the fiber bundle becomes trivial bundle

$$Z \times \tilde{S} \to \tilde{S},$$

clearly the morphism is projective and admits an relative ample line bundle L (since Z is projective). And therefore if we define

$$L' = \bigotimes_{g\in\Gamma} g^* L.$$

Since L' is monodromy invariant, the ample line bundle will descend to the original fiber bundle $g : X \to S$ and thus g is a projective morphism.

For (1) implies (3). Since $g : X \to S$ is Moishezon, by Definition 3.9, there exists a g-big (rank 1 reflexive sheaf) H on X (since it's fiber bundle so the restriction on Z is again big and denote it also as H). Given an ample line bundle L on the fiber Z, we consider the monodromy action on L, which pulls back the ample line bundle to another ample line bundle $L_\gamma = \rho(\gamma)^*L$.

Note that under the monodromy action, the intersection

$$d := H \cdot (L_\gamma)^{n-1},$$

remain the same for all γ.

We then consider the linear functional

$$\ell : N^1(Z)_{\mathbb{R}} \to \mathbb{R}, \quad M \mapsto M^{n-1} \cdot H,$$

if we restrict the linear functional on the ample cone $\mathrm{Amp}(Z)$, then

$$S_d = \{M \in N^1(Z)_{\mathbb{R}} \cap \mathrm{Amp}(Z) \mid M^{n-1} \cdot H = d\},$$

is a bounded slice. To see this, by the Khovanskii-Teissier inequality we have

$$\left(H \cdot M^{n-1}\right)^n \geq (H^n)\,(M^n)^{n-1},$$

thus we get

$$\mathrm{vol}(M) \leq \left(\frac{d^n}{H^n}\right)^{\frac{1}{n-1}},$$

so that it's bounded (for a fixed H). Thus the slice contains only finite many lattice points of $\mathrm{NS}(Z)$.

$$\#\{M \in \mathrm{NS}(Z) \cap \mathrm{Amp}(Z) \mid \ell(M) = d\} < \infty.$$

In particular, the ample line bundle on the monodromy orbit is finite

$$\Gamma \cdot L = \{L_\gamma := \rho(\gamma) \cdot L \mid \gamma \in \pi_1(S)\} \subset \{M \in \mathrm{NS}(Z) \cap \mathrm{Amp}(Z) \mid \ell(M) = d\}.$$

This will force the monodromy to be finite, indeed apply the orbit-stablizer theorem

$$\boxed{|\Gamma| = |\Gamma \cdot L||\mathrm{Stab}(L)| < +\infty}$$

Thus only needs to prove the $\mathrm{Stab}(L) = \mathrm{Aut}(Z, L) = \{\phi \in \mathrm{Aut}(Z) \mid \phi^*L = L\}$ is finite. On the other hand, since $\mathrm{Aut}(Z)$ is discrete, by Proposition 3.23, this means that $\mathrm{Stab}(L) = \mathrm{Aut}(Z, L)$ is finite. □

4 Examples

In this section, we will present varies examples related to the Moishezon space and Moishezon morphism.

4.1 The Hironaka's example

Hironaka discovered a bunch of complete non-projective 3-fold which is called Hironaka's varieties. Note that based on the construction of Hironaka, we can from almost all the projective varieties construct some Moishezon spaces, that is why we said at the beginning that Moishezon spaces are versatile in birational geometry. (However, this is not true in dimension 2, since all the smooth Moishezon surface are actually projective, see e.g. [GPR94]). The major reference of this part of note is the paper by Ulrich Thiel (see `https://ulthiel.com/math/wp-content/uploads/other/hironakas_example.pdf`).

Given a smooth projective threefold, which contains two rational curves transversely intersection at two points. Assume that two rational curves are C and D that intersect at the point P, Q.

We then take two steps, blow up

$$X_1 = \mathrm{Bl}_{(D\backslash P)'}\left(\mathrm{Bl}_{C\backslash P}(X\backslash P)\right) \xrightarrow{\pi_2} \mathrm{Bl}_{C\backslash P}(X\backslash P) \xrightarrow{\pi_1} X\backslash P$$
$$X_2 = \mathrm{Bl}_{(C\backslash Q)'}\left(\mathrm{Bl}_{D\backslash Q}(X\backslash Q)\right) \xrightarrow{\sigma_2} \mathrm{Bl}_{D\backslash Q}(X\backslash Q) \xrightarrow{\sigma_1} X\backslash Q,$$

Note that if we define $U = X - \{P, Q\}$, then $\pi^{-1}(U) \cong \sigma^{-1}(U)$. In particular, we can glue X_1 and X_2 along $\pi^{-1}(U)$ and $\sigma^{-1}(U)$. In the picture below, we glue the red exceptional surface on the right hand side with the black exceptional surface on the left hand side (denote it S_1) and the blue exceptional surface on the left hand side with the black exceptional surface on the right hand side (denote it S_2). (see pictrue 2). By the gluing lemma, there exists a morphism $f : H \to X$ and the restriction of the morphism on S_1, S_2 as $f_1 = f|_{S_1} : S_1 \to C$ and $f_2 = f|_{S_2} : S_2 \to C$.

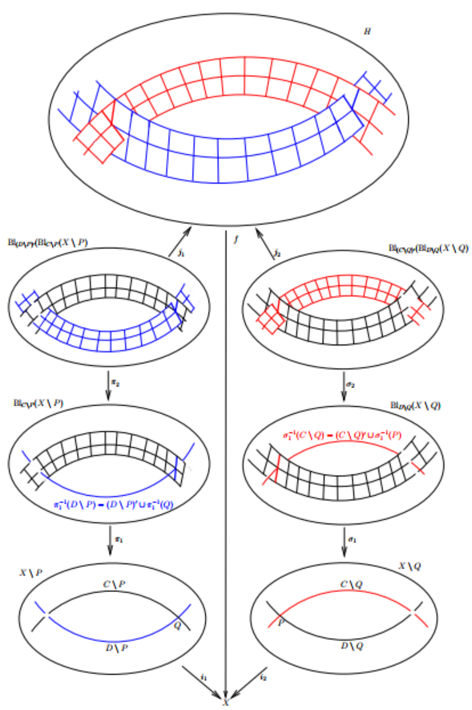

Figure 2: Construction of the Hironaka's variety

We claim that Hironaka's variety is non-projective. The idea to prove the non-projective is to find some curve on the surface $S = S_1 \cup S_2$ which has positive degree but add up to 0.

The key observation is that $f^{-1}(P)$ (resp. $f^{-1}(Q)$) decompose into two split projective lines L_Q and L'_Q in S_1 (resp. L_P and L'_P in S_2). (see the precise statement below).

Choose two points $A \in C - \{P, Q\}$ and $B \in D - \{P, Q\}$. Since all the points on a rational curve are linear equivalent, therefore

$$A \sim_C Q \Longrightarrow f_1^{-1}(A) \sim_{S_1} f_1^{-1}(Q) = L_Q + L'_Q$$
$$B \sim_D P \Longrightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(P) = L_P + L'_P$$

and Push forward of cycle, we get equivalence on S.

$$I : f^{-1}(A) \sim_S f^{-1}(Q) = L_Q + L'_Q$$
$$II : f^{-1}(B) \sim_S f^{-1}(P) = L_P + L'_P$$

On the other hand we also that B, Q lies in the same rational curve, so that

$$III : B \sim_D Q \Rightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(Q) \Rightarrow f^{-1}(B) \sim_S L'_Q$$

and combined then together, we get

$$\begin{aligned} f^{-1}(A) + f^{-1}(B) \sim_S f^{-1}(A) + f^{-1}(B) &\Rightarrow L_Q + L'_Q + L_P + L'_P \sim_S L'_Q + L'_P \\ &\Rightarrow L_Q + L_P \sim_S 0 \end{aligned}$$

If there exist some ample divisor on A, then both $L_Q \cdot A > 0$ and $L_P \cdot A > 0$ contradict the linearly trivial relation above. Therefore the only possible case is Hironaka's variety is non-projective.

4.2 Flop the lines on general quntic threefold produce Moishezon variety

4.3 Locally Moishezon morphism which is not globally Moishezon

There are rational and K3 surfaces with infinite, discrete automorphism group. These lead to fiber bundles over the punctured disc $\mathbb{D}^\circ$ that are locally Moishezon but not globally Moishezon (using the Proposition 3.22).

4.4 Singular Kähler Moishezon space needs not to be projective

By blowing down elliptic curves, such an easy example is not possible. Instead, consider a cubic $C \subset \mathbb{P}_2$ and let $x_1, \ldots, x_{10}$ be general points on C. Let $f : X \to \mathbb{P}_2$ be the blow-up of these point. Then the strict transform $\hat{C}$ of C in X is elliptic with $\hat{C}^2 = -1$. It can be shown that the blow-down of $\hat{C}$ is not projective.

4.5 Fiberwise projective morphism needs not to be projective morphism

Let $S_0 := (g = 0) \subset \mathbb{P}^3_{\mathrm{x}}$ and $S_1 := (f = 0) \subset \mathbb{P}^3_{\mathrm{x}}$ be surfaces of the same degree. Assume that S_0 has only ordinary nodes, S_1 is smooth $\mathrm{Pic}\,(S_1)$ is generated by the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ and S_1

does not contain any of the singular points of S_0. Fix $m \geq 2$ and consider

$$X_m := (g - t^m f = 0) \subset \mathbb{P}^1_{\mathbf{x}} \times \mathbb{A}^1_t.$$

The singularities are locally analytically of the form $xy+z^2-t^m = 0$. Thus X_m is locally analytically factorial if m is odd. If m is even then X_m is factorial since the general fiber has Picard number 1, but it is not locally analytically factorial; blowing up $\big(x = z - t^{m/2} = 0\big)$ gives a small resolution. Thus we get that (4.1) X_m is bimeromorphic to a proper, smooth family of projective surfaces iff m is even, but (4.2) X_m is not bimeromorphic to a smooth, projective family of surfaces.

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Fiberwise Bimeromorphic Problems Summer 2025

Note 2 — 2025-07-09 (draft version)

Yi Li

1 Overview

The aim of this note is to study the fiberwise bimeromorphic problems. To be more concrete, we consider the following two problems. The first one is:

Question 1.1. Let

$$\pi : X \to B \text{ and } \pi' : X' \to B,$$

be proper flat morphisms from a complex analytic space to a smooth connected curve B. Assume that the generic fibers of π and π' are bimeromorphic. Under what conditions, the special fibers between these two families also admit a certain bimeromorphic relation?

The second one focuses on the Moishezon morphisms, under which condition we can let a Moishezon morphism fiberwise bimeromorphic to a projective morphism:

Question 1.2. Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Under which conditions, it is actually fiberwise bimeromorphic to a projective morphism $g^p : X^p \to \mathbb{D}$?

We will discuss the first question in Sections 2–4 and the second question in Section 5. The main references for this note are [Kol22] and [KT19].

Contents

2 A Fiberwise Birational Criterion

We first recall the definition of the meromorphic S map that we introduced in the first time.

Definition 2.1 (Meromorphic S-map)**.** Let X, Y be reduced complex spaces. We call the S-map a *meromorphic S-map*

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

if there exists a subvariety $\Gamma \subset X \times_S Y$ with the restriction of the first projection $p : \Gamma \to X$ be a proper bimeromorphic morphism. Moreover, if the restriction on the second projection $q : \Gamma \to Y$ is also a proper bimeromorphic morphism, then we call α *proper bimeromorphic S-map*.

Definition 2.2 (Fiberwise bimeromorphic map, [Kol22, Definition 26])**.** Let $g_i : X^i \to S$ be a proper morphisms. A bimeromorphic S-map $\phi : X^1 \dashrightarrow X^2/S$ is *fiberwise bimeromorphic* if ϕ induces a bimeromorphic map $\phi_s : X^1_s \dashrightarrow X^2_s$ for every $s \in S$.

Remark 2.3 (Fiberwise bimeromorphic $\neq$ fibers bimeromorphic equivalent, [CRT25, Example 2.15])**.** Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ be the Hirzebruch surface of index n. By construction easy to see that all the Hirzebruch surface are birational equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$. Let Z be any compact complex manifold. So that $\mathbb{F}_n \times Z \to \mathbb{P}^1$ is binational equivalent to $(\mathbb{P}^1 \times \mathbb{P}^1) \times Z \to \mathbb{P}^1$.

$$\begin{tikzcd} & \mathrm{Bl}_{p_1, \cdots, p_n}(\mathbb{P}^1 \times \mathbb{P}^1) \times Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^1 \times \mathbb{P}^1 \times Z \arrow[rr, dashed, "q \circ p^{-1}"] \arrow[dr] & & \mathbb{F}_n \times Z \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Note that fibers of these two families are birational equivalent (as both side have fiber $\mathbb{P}^1 \times Z$). However the restriction of the map $q \circ p^{-1}$ does not give the bimeromorphic map of the fiber (since the strict transform of the fibers of $\mathbb{P}^1 \times \mathbb{P}^1 \times Z$ via p^{-1} will be contracted by q).

Although the bimeromorphic map needs not to be fiberwise bimeromorphic in general, it is indeed fiberwise bimeromorphic on a dense open subset. As the following proposition shows.

Proposition 2.4 (Bimeromorphic on the generic fiber)**.** Let $f : X \dashrightarrow Y$ be a bimeromorphic S-map between two proper surjective family $g : X \to S,\ h : Y \to S$ over the base S. Then on the generic fiber, the map f will induce a bimeromorphic $f_s : X_s \dashrightarrow Y_s$.

Proof. Since f is bimeromorphic, by definition, the graph $\Gamma \subset X \times_S Y$ will induce two bimeromorphic morphisms

$$p : \Gamma \to X, \quad q : \Gamma \to Y,$$

such that there exists some non-empty analytic Zariski open subset $U_X \subset X$, $U_Y \subset Y$ with $p : p^{-1}(U_X) \to U_X$, $q : q^{-1}(U_Y) \to U_Y$ be isomorphisms. On the generic fiber, the dimension equalities hold

$$\dim X_s = \dim X - \dim S, \quad \dim Y_s = \dim Y - \dim S.$$

On the other hand, if we denote the analytic subset $E_X = X - U_X$ and $E_Y = Y - U_Y$, then the intersection with the generic fiber X_s (resp. Y_s), say

$$E_X \cap X_s = E_{X,s}, \text{ (resp. } E_Y \cap Y_s = E_{Y,s}),$$

are proper analytic subset in X_s (resp. Y_s). Indeed, only needs to show that $\dim E_{X,s} < \dim X_s$ (resp. $\dim E_{Y,s} < \dim Y_s$). As intersection of analytic subvariety is still analytic subvariety and dimension strict less, it's automatically proper analytic subset. Thus by definition $p_s : \Gamma_s \to X_s$ (resp. $q_s : \Gamma_s \to Y_s$) are bimeromorphic morphisms. To see that $\dim E_{X,s} < \dim X_s$, we divide it into two cases: (1) If $g(E_X) \subset S$ is proper analytic subset, then clearly the generic fiber has $\dim E_{X,s} = 0$. (2) If $g(E_X) = S$ then the generic fiber $\dim E_{X,s} = \dim E_X - \dim S$ and we know that $\dim E_X < \dim X$ and therefore

$$\dim E_{X,s} = \dim E_X - \dim S < \dim X - \dim S = \dim X_s.$$

Since the base change preserves the properness, we have $p_s : \Gamma_s \to X_s$, $q_s : \Gamma_s \to Y_s$ are still proper. Thus, complete the proof. $\square$

We now prove the first main theorem of this note, which is about the specialization of the birational map when the pluricanonical system is non-empty.

Proposition 2.5 (Kollár's Specialization of birational map, [Kol23, Proposition 1.25]). Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the generic fibers X_b^1 and X_b^2 (for $b \neq 0$) are birational, and further assume that the pluricanonical system $\left|mK_{X_b^i}\right|$ is non empty for some $m > 0$. Then for every $b \in B$, the fibers X_b^1 and X_b^2 are birational.

Proof. Pick a birational map $\phi : X_b^1 \dashrightarrow X_b^2$ (for the generic fiber), and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the resolution of the graph with projections $p_i : Y \to X^i$.

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
 & \Gamma \arrow[dl] \arrow[dr] & \\
X^1 \arrow[dr, "f_1"'] & & X^2 \arrow[dl, "f_2"] \\
 & B &
\end{tikzcd}$$

Note that by definition, both of the p_i are open embeddings on $Y \setminus (\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$.

Thus if we prove that neither $p_1\,(\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$ nor $p_2\,(\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$ contains a fiber of f_1 or f_2, then $p_2 \circ p_1^{-1} : X^1 \dashrightarrow X^2$ (it needs not to be birational) restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$.

We may assume that B is affine (as we only care about the special fiber, thus we can focus on the affine base around b) and let $\mathrm{Bs}\,|mK_{X^i}|$ denote the set-theoretic base locus. Let $L_i = \mathcal{O}_{X^i}(mK_{X^i})$. The direct image $\mathcal{E}_i = f_{i*}L_i$ as a torsion free sheaf is locally free on the smooth curve B (so that the vanishing locus of a section of $\mathcal{E}_i$ is a subvariety).

By assumption $|mK_{X^i_b}| \neq \emptyset$ for generic $b \in B$, we claim that $|mK_{X^i}|$ is non-empty as well. Indeed, since the restriction map

$$H^0(X^i, mK_{X^i}) \to H^0(X^i_b, mK_{X^i_b}),$$

is surjective on the generic fibers. Thus, the pluricanonical system on the generic fiber satisfies $|mK_{X^i}|_{X^i_b} = |mK_{X^i_b}| \neq \emptyset$. In particular, this means that $|mK_{X^i}| \neq \emptyset$.

On the other hand, since $H^0(X, mK_{X^i}) \neq 0$ and the base is affine, for any point $s \in B$, there exists a non-zero section

$$0 \neq \sigma \in H^0(B, \mathcal{E}_i),$$

such that $\sigma(s) \neq 0$. Therefore consider the restriction commutative diagram (note that in general it's not clear the base change morphism $H^0(X^i_s, L_i|_{X^i_s}) \to \mathcal{E}_i(s)$ is isomorphism or not on the special fiber)

$$\begin{tikzcd}
H^0(X^i, L_i) \arrow[r, "\mathrm{res}"] \arrow[d, "\simeq"'] & H^0(X^i_s, L_i|_{X^i_s}) \arrow[d] \\
H^0(B, \mathcal{E}_i) \arrow[r, "\mathrm{res}"'] & \mathcal{E}_i(s)
\end{tikzcd}$$

there exists a section $\sigma' \in H^0(X^i, L_i)$ which maps down to $\sigma \in H^0(B, \mathcal{E}_i)$ such that $\sigma(s) \neq 0$. So that $\sigma'|_{X^i_s} \neq 0$ in $H^0(X^i_s, L_i|_{X^i_s})$. And therefore the base locus $\mathrm{Bs}|mK_{X^i}|$ cannot contain the fiber.

Since X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \qquad \text{where } E_i \geq 0 \text{ and } \operatorname{Supp} E_i = \mathrm{Ex}\,(p_i)\,.$$

So that every section of $\mathcal{O}_Y\,(mK_Y)$ pulls back from X^i, Thus

$$\mathrm{Bs}\,|mK_Y| = p_i^{-1}\,(\mathrm{Bs}\,|mK_{X^i}|) + \operatorname{Supp} E_i,$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}\,(\mathrm{Bs}\,|mK_{X^1}|) + \operatorname{Supp} E_1 = p_2^{-1}\,(\mathrm{Bs}\,|mK_{X^2}|) + \operatorname{Supp} E_2,$$

Therefore,

$$\boxed{p_1\,(\operatorname{Supp} E_2) \subset p_1\,(\operatorname{Supp} E_1) + \mathrm{Bs}\,|mK_{X^1}|}$$

Since E_1 is p_1-exceptional, $p_1\,(\operatorname{supp} E_1)$ has codimension ≥ 2 in X^1, hence it does not contain any of the fibers of f_1. Combined with $\mathrm{Bs}\,|mK_{X^1}|$ does not contain any of the fibers either.

Thus, $p_1\left(\operatorname{Ex}(p_1)\cup\operatorname{Ex}(p_2)\right)$ does not contain any of the fibers, and the same argument shows for $p_2\left(\operatorname{Ex}(p_1)\cup\operatorname{Ex}(p_2)\right)$. $\square$

As a remark by [Kol23], the result holds true even when the pluricanonical systems are empty. That is what we will prove in the next section.

3 Kontsevich-Tschinkel's Fiberwise Birational Theorem

Theorem 3.1 ([KT19, Theorem 1]). Let

$$\pi : X \to B \text{ and } \pi' : X' \to B$$

be smooth proper morphisms to a smooth connected curve B, over a field of characteristic zero. Assume that the generic fibers of π and π' are birational over the function field of B. Then, for every closed point $b \in B$, the fibers of π and π' over b are birational over the residue field at b.

We first introduce some new notions that needed in the proof.

Definition 3.2 (semi-ring). A *semi-ring* $(S, +, \times)$ consists of a set S equipped with two binary operations $+, \times$. Such that $+$ makes S a commutative monoid (which does not need to be an Abelian group compared to the definition of a ring).

Definition 3.3 (Burnside semi-ring over a field k, [KT19, Definition 2]). The *Burnside semi-ring* $\mathrm{Burn}_+(k)$ of a field k is the set of $\sim_k$ equivalence classes of smooth schemes of finite type over k endowed with a semi-ring structure where multiplication and addition are given by disjoint union and product over k. (here the $\sim_k$ equivalence of two schemes X, X' are defined as follows: $X/k \sim_k X'/k$ if and only if X and X' are k-birational). To be more precise, the addition and multiplication of semi-ring structure is defined as follows:

(a) Addition: Disjoint union $[X] + [Y] = [X \sqcup Y]$.
(b) Multiplication: Cartesian product $[X] \cdot [Y] = [X \times Y]$.

We then introduce the Grothendieck ring, and we denote $\mathrm{Burn}(k)$ the Grothendieck ring generated by $\mathrm{Burn}_+(k)$.

Definition 3.4 (The Grothendieck ring $\mathrm{Burn}(k)$). The *Grothendieck ring* $\mathrm{Burn}(k)$ thhat is associated to the Bunrside semi-ring $\mathrm{Burn}(k)^+$ is defined as the set of equivalence classes of pairs $([X], [Y])$, where $[X], [Y] \in \mathrm{Burn}(k)^+$. Intuitively, $([X], [Y])$ represents the "difference" $[X] - [Y]$. With the equivalence relation: We say $([X], [Y]) \sim ([X'], [Y'])$ if there exists $[Z] \in \mathrm{Burn}(k)^+$ such that:

$$[X] + \left[Y'\right] + [Z] = \left[X'\right] + [Y] + [Z].$$

The ring Operations is defined as follows

(a) Addition: $([X], [Y]) + ([X'], [Y']) = ([X] + [X'], [Y] + [Y'])$.
(b) Multiplication: $([X], [Y]) \cdot ([X'], [Y']) = ([X \times X'] + [Y \times Y'], [X \times Y'] + [Y \times X'])$.

Remark 3.5. The reason to introduce the Grothendieck ring over the Burnside semi-ring is that it allows one to implement formal subtraction, cut and paste operations.

Remark 3.6. Note that we can decompose the

$$\mathrm{Burn}(k) = \sqcup_{n\geq 0} \mathrm{Bir}_n(k),$$

where $\mathrm{Bir}_n(k)$ denotes k-birational equivalent class of smooth variety of dimension n. Each class can be denoted by $[L/k]$ with $L = k(X)$.

Proposition 3.7 (Existence of SNC model)**.** Let R be a complete dvr, let K be the fractional field (i.e. the generic point of $\mathrm{Spec}(R)$) and k the residue field (i.e. the special point of $\mathrm{Spec}(R)$). Let X/K be a geometric connected smooth proper variety defined over K, then there exists a regular flat separated R-scheme of finite type $\mathscr{X}$, endowed with an isomorphism of K-scheme $\mathscr{X}_K \to X$ such that the special fiber $\mathscr{X}_k$ is a divisor with strict normal crossing. We call $\mathscr{X}/R$ is a SNC model of X/K.

Proof. Let us first briefly sketch out the idea. We first reduce the problem to the projective case. □

Remark 3.8. The SNC model also plays an important role in the

Definition 3.9 (Specialization map, [KT19, (3.2)])**.** Let $\mathfrak{o}$ be a complete dvr, let K be the fractional field (i.e. the generic point of $\mathrm{Spec}(\mathfrak{o})$) and k the residue field (i.e. the special point of $\mathrm{Spec}(\mathfrak{o})$). We define

$$\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}\left[\mathrm{Bir}_n(k)\right],$$

as follows: given a smooth projective family $X_K \to \mathrm{Spec}(K)$ (with the function field $L := K(X_K)$), choose one of family

$$\pi : X \to \mathrm{Spec}(\mathfrak{o}),$$

where π is proper, such that the generic fibers is X_K and special fiber

$$X_0 = \bigcup_{i\in I} d_i D_i,$$

is a SNC divisor, with the strata $D_J := \bigcap_{j\in J} D_j$. We then define the specialization map to be

$$\boxed{\rho_n([L/K]) := \sum_{\emptyset\neq J\subseteq I} (-1)^{\#J-1}\left[D_J \times \mathbb{A}^{\#J-1}/k\right]}$$

One of the main difficulties in the proof is verifying that the specialization map $\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}[\mathrm{Bir}_n(k)]$ is well-defined (i.e., it does not depend on the choice of the family $X \to D$) or representative X in $\mathrm{Bir}_n(X)$. We omit the proof of this part; for details, see [KT19, Theorem 4]).

Proof of Theorem 3.1. We first reduce the problem onto Spec of a complete dvr. Let $\pi : X \to B$ be a smooth proper morphism to a smooth connected curve B over k with fiber X over the generic point of B. Let $K = k(B)$ be the function field of B. Let κ_b be the residue field at b, a finite extension of k. Let K_b be the completion of K at b. Then K_b is a local field with residue field κ_b. Let

$$\phi_b : K \to K_b,$$

be the canonical inclusion. By functoriality, it defines a homomorphism

$$\phi_{b,*} : \mathrm{Burn}(K) \to \mathrm{Burn}\left(K_b\right).$$

We then consider the specialization map over the complete dvr K_b. Note that we have the specialization homomorphism

$$\rho : \mathrm{Burn}\,(K_b) \to \mathrm{Burn}\,(\kappa_b)\,,$$

and the following identity

$$[X_b/\kappa_b] = \rho\left(\phi_{b,*}([X/K])\right),$$

which follows immediately from the Definition 3.9 of ρ, since the special fiber is smooth and irreducible. This shows that the birational type of the special fiber is determined by the birational type of the generic fiber. ☐

4 Fiberwise Bimeromorphic Criterion using Plurigenera

In this section, we will give a criterion for fiberwise bimeromorphic map using plurigenera. For readers who want to know more about this, please refer to [CRT25].

Lemma 4.1 ([GPR94, Theorem 1.19])**.** Let $f : X \to Y$ be a proper surjective holomorphic map, assume that X is reduced and irreducible. Then the set

$$\{y \in Y \mid \dim_x X_y > \dim X - \dim Y \text{ for some } x \in X_y\}\,.$$

is analytic in Y and of codimension at least 2.

Proposition 4.2 ([CRT25, Theorem 1.4])**.** Let X, Y and S be complex analytic spaces. Assume that X is reduced (not necessarily normal) and irreducible, Y is normal, and S is a smooth curve. Assume further that both $\pi_1 : X \to S$ and $\pi_2 : Y \to S$ are proper surjective holomorphic maps. Suppose that there is a bimeromorphic morphism $f : X \to Y$ over S. For some $t \in S$, if D_t is an irreducible component of Y_t that is of codimension 1 in Y, then there exists an irreducible component C_t (equipped with the reduced structure) of X_t that is bimeromorphic to D_t, induced by f.

In particular, if the fibers of $X \to S$ and $Y \to S$ are irreducibles then f is fiberwise bimeromorphic map.

Proof. Since X is reduced and irreducible, by lemma above, we have the set of points that $\dim X_y = 0$ is a big open subset in Y (with the complement an analytic subset V such that $\mathrm{codim}_V(X) \geq 2$). Since Y is normal, and $f : X - f^{-1}(V) \to Y - V$ is bijective. Thus $f : X - f^{-1}(V) \to Y - V$ is biholomorphic. Additionally, f is surjective by the definition of a bimeromorphic morphism. Consequently, there exists an irreducible component C_t of X_t such that $f\,(C_t) = D_t$ by the irreducibility of D_t.

In view of the codimensions of V and D_t, it follows that $D_t \nsubseteq V$, and consequently, $C_t \nsubseteq f^{-1}(V)$. Clearly, $D_t \cap V$ is a thin analytic subset of D_t, and $C_t \cap f^{-1}(V)$ is a thin analytic subset of C_t. Hence, one can easily check by definition that $f : C_t \to D_t$ is bimeromorphic. ☐

We next prove a simplify version of the fiberwise bimeromorphic cirterion using plurigenera, for a much more general version, please refer to [CRT25].

Theorem 4.3 ([CRT25, Theorem 1.6]). Let

$$\pi_1 : X \to S,\ \pi_2 : Y \to S$$

be two (locally) Moishezon morphism with irreducible fibers that admits canonical singularities, such that $\kappa(X_0) \geq 0$. Then the bimeromorphic map that connects π_1 and π_2 is indeed fiberwise bimeromorphic.

Let us briefly sketch out the idea. We first take the resolution of indeterminacy, by further resolution we can guarantee the generic fibers of $W \to S$ being smooth.

$$\begin{tikzcd}
 & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\
 & S &
\end{tikzcd}$$

We claim that the strict transform $\tilde{X}_0 = p_*^{-1}(X_0)$ and the strict transform $\tilde{Y}_0 = q_*^{-1}(Y_0)$ must coincide. For otherwise, since plurigenera is bimeromorphic invariant we have $P_m(X_0) = P_m(\tilde{X}_0)$, $P_m(Y_0) = P_m(\tilde{Y}_0)$ and $P_m(W_t) = P_m(X_t) = P_m(Y_t)$. On the other hand, since the family $W \to S$ is Moishezon, by the lower semi-continuity of the plurigenera (for Moishezon morphism), we have $P_m(\tilde{X}_0) + P_m(\tilde{Y}_0) \leq P_m(W_t)$. Since $Y \to S$ be a Moishezon morphism with fiberwise canonical singularities, the plurigenera remain constant i.e. $P_m(Y_t) = P_m(Y_0)$. Putting those together, we have

$$\boxed{P_m(X_0) + P_m(\tilde{Y}_0) = P_m(\tilde{X}_0) + P_m(\tilde{Y}_0) \leq P_m(W_t) = P_m(X_t) = P_m(Y_t) = P_m(Y_0) = P_m(\tilde{Y}_0)}$$

so that the plurigenera $P_m(X_0) = 0$ which contradicts $\kappa(X_0) \geq 0$.

5 The Fiberwise Bimeromorphic Conjecture for Moishezon Morphisms

In the last section, we will prove the following conjecture under the additional assumption that the center fiber is KLT and not uniruled.

Conjecture 5.1 (Fiberwise bimeromorphic conjecture for Moishezon morphism, [Kol22, Conjecture 5]). Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 has canonical (resp. log terminal) singularities.

Then g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ such that

(1) X_0^{p} has canonical (resp. log terminal) singularities,
(2) X_s^{p} has terminal singularities for $s \neq 0$, and
(3) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.

Remark 5.2. Before continuing our discussion of this conjecture, let us first look closely at what this conjecture is about. The conjecture shows that the flat Moishezon morphism is not only bimeromorphic to some projective model but it is indeed fiberwise bimeromorphic to some projective model, as long as the singularity on the central fiber is nice enough.

Kollár verifies the conjecture when the central fiber is KLT and not uniruled. Before proving the theorem, let us list some intermediate results that will be used.

Proposition 5.3 (Inversion of adjunction, [Kol22, Proposition 30])**.** Let X be a normal complex analytic space, $X_0 \subset X$ a Cartier divisor, and Δ an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then $(X, X_0 + \Delta)$ is PLT in a neighborhood of X_0 iff $\left(X_0, \Delta|_{X_0}\right)$ is KLT.

Proposition 5.4 (Existence of canonical modification, [Kol22, Corollary 38])**.** Let $f : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 is log terminal. Then X has a canonical modification $\pi : X^{\mathrm{c}} \to X$, such that
(a) X_0^{c} is log terminal and,
(b) π is fiberwise birational.

Proof. The proof uses some algebraic approximation technique, see [Kol22]. □

Lemma 5.5 ([Kol22, Lemma 31.1])**.** Let $X \to S$ be a proper, Moishezon morphism, D an $\mathbb{R}$-divisor on X, and A a big $\mathbb{R}$-divisor on X such that $\mathbf{B}^{\mathrm{div}}(A) = \emptyset$. Then, for every prime divisor $F \subset X$,

$$\operatorname{coeff}_F \mathbf{B}_-^{\mathrm{div}}(D) = \lim_{\epsilon \to 0} \operatorname{coeff}_F \mathbf{B}_-^{\mathrm{div}}(D + \epsilon A)$$

Lemma 5.6 ([Kol22, Lemma 31.2])**.** Let $X_i \to S$ be proper, Moishezon morphisms, $h : X_1 \to X_2$ a proper, bimeromorphic morphism, D_2 a pseudo-effective, $\mathbb{R}$-Cartier divisor on X_2, and E an effective, h-exceptional divisor. Then

$$\mathbf{B}_-^{\mathrm{div}}\left(E + h^* D_2\right) \geq E.$$

The following proposition is useful in the proof.

Proposition 5.7. Let $f : X \to U$ be a proper morphism between complex varieties, (X, Δ) a DLT pair and $\phi : X \dashrightarrow X_M$ be a minimal model for $K_X + \Delta$ over U. Then the set of ϕ-exceptional divisors coincides with the set of divisors contained in $\mathbf{B}_-\left(K_X + \Delta/U\right)$.

Proof. Let $p : Y \to X$ and $q : Y \to X_M$ be a common resolution. Since ϕ is $(K_X + \Delta)$-negative, we have that $p^*\left(K_X + \Delta\right) = q^*\left(K_{X_M} + \phi_*\Delta\right) + E$ where E is effective, q-exceptional and the support of p_*E is the set of ϕ-exceptional divisors. Since the minimal model assumption, we have $N_\sigma\left(p^*\left(K_X + \Delta\right)/U\right) = E$. we get

$$p_*E = N_\sigma(K_X + \Delta).$$

□

Lemma 5.8. Let $b_0 = 1, b_1, \ldots, b_n$ be real numbers which are linearly independent over $\mathbb{Q}$, and suppose that the divisor $\sum_{i=0}^n b_i B_i$ is $\mathbb{R}$-Cartier. Then each of the divisors B_i is $\mathbb{Q}$-Cartier.

Having introduced a bunch of lemma will be used in the proof. We can now dive into the proof of the last main theorem of this note.

Theorem 5.9 (A flat Moishezon morphism with KLT and non-uniruled central fiber will be fiberwise bimeromorphic to a projective morphism, [Kol22], Theorem 28)**.** Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that

1. X_0 has log terminal singularities and
2. X_0 is not uniruled

Then

(a) g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ (possibly over a smaller disc),

(b) X_0^{p} has log terminal singularities,

(c) X_s^{p} is not uniruled and has terminal singularities for $s \neq 0$,

(d) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier

Proof. We take a resolution of singularities $Y \to X$ such that $Y \to \mathbb{D}$ is projective, and then take a relative minimal model of $Y \to \mathbb{D}$. We hope that it gives what we want. There are, however, several obstacles.

Step 1. Take the canonical modification. We need to control the singularities of X. First for a flat proper Moishezon morphism with KLT central fiber, there exist a canonical modifiction (Theorem 5.4) which is fiberwise birational and the central fiber is KLT. Thus we in the case that K_X is $\mathbb{Q}$-Cartier.

Indeed by the canonical modificaiton we can find some canonical modification $X^c \to X$ such that X^c is canonical singularity and the the morphism $X^c \to X$ is the fiberwise birational. Thus, if we can prove the result for $X^c \to \mathbb{D}$ then it will also be true for the $X \to \mathbb{D}$ (since composition of fiberwise birational map is again fiberwise birational).

We assume this from now on. Then the inversion of adjunction for PLT pair implies that the pair (X, X_0) is PLT. by setting $\Delta = 0$ in the inversion of adjunction. (To apply the inversion of adjunction here we require K_X to be $\mathbb{Q}$-Cartier)

Step 2. Take the semi-stable reduction. After a base change $z \mapsto z^r$ we get $g^r : X^r \to \mathbb{D}$. For suitable r, there is a semi-stable, projective resolution $h : Y \to \mathbb{D}$; we may also choose it to be equivariant for the action of the cyclic group $G \cong \mathbb{Z}_r$. All subsequent steps will be G-equivariant. We denote by X_0^Y the birational transform of X_0 and by E_i the other irreducible components of Y_0.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X^r \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathbb{D} \arrow[r] & \mathbb{D} \end{tikzcd}$$

Such that the following conditions hold:
(a) Y is non-singular,

(b) generic fibers are non-singular,
(c) The special fiber is a reduced divisor with SNC support,
(d) Denote that $Y_0 = X_0^Y + \sum c_i E_i$ (with X_0^Y be the strict transform on X_0), note that the strict transform X_0^Y will dominant X_0.

Step 3. Prove the generic fibers Y_s are not uniruled (for $s \neq 0$). We will prove it by contradiction, if the generic fibers Y_s are uniruled. Then, by Matsusaka's theorem (see [Kol96, Theorem VI.1.7]), all the irreducible components of Y_0 are uniruled. On the other hand, since X_0^Y dominant X_0, X_0 must be uniruled, a contradiction.

And finally by the BDPP theorem. easy to see K_{Y_s} is pseudo-effective. (Since we assume that generic fibers are smooth).

Step 4. Run the MMP. We require the condition that the general fibers are of log general type. To achieve this, let H be an ample, G-equivariant divisor such that $Y_0 + H$ is snc (note that this is possible by taking $H' = \otimes_{m=1}^{n} g \cdot H$, since G is finite group this is well defined ample line bundle). For $\epsilon > 0$ we get a pair $(Y, \epsilon H)$ whose general fibers $(Y_s, \epsilon H_s)$ are of log general type since K_{Y_s} is pseudoeffective by previous step. For such algebraic families, relative minimal models are known to exist by BCHM. We also know that $(Y, Y_0 + \epsilon H)$ is dlt for $0 < \epsilon \ll 1$ (since Y is smooth and $Y_0 + H$ is snc).

Thus we get the $(K_X + Y_0 + \epsilon H)$-relative MMP on the disc $\mathbb{D}$, (Note that the base is an analytic disc, thus the MMP is in the sense of Fujino [Fuj22] or Kollár-Nicaise-Xu [KNX18]).

$$\begin{tikzcd} (Y, \epsilon H) \arrow[rr, dashed, "\phi"] \arrow[dr] & & (Y^m, \epsilon H^m) \arrow[dl] \\ & \mathbb{D} & \end{tikzcd}$$

We claim $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}})$ is DLT, and H^{m} is $\mathbb{Q}$-Cartier for general choice of ϵ and also thus $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is also dlt.

Indeed, Step of MMP will preserve DLT condition (see [**BCHM**] Lemma 3.10.10.) easy to see $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}})$ is DLT. On the other hand, by Lemma 5.8, easy to see if ϵ is sufficient general the $\mathbb{Q}$-linear independent condition satisfies and therefore H^m is indeed a $\mathbb{Q}$-Cartier divisor. And finally by [KM98, Corollary 2.39] the $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is also DLT.

Recall that we have
$$\mathbf{B}_{-}^{\mathrm{div}}(K_Y + Y_0) \geq (1 + a(E_i, X^r, X_0))E_i,$$
since the discrepancy of a PLT pair $a(E_i, X^r, X_0) > -1$ thus all the exceptional divisors E_i contains in the divisorial part of the restricted base locus $\mathbf{B}_{-}^{\mathrm{div}}(K_Y + Y_0)$. On the other hand

$$\operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D) = \lim_{\epsilon \to 0} \operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D + \epsilon A),$$

for any prime divisor F. Thus, for sufficiently small ϵ, E_i also contains in the restricted base locus of $K_Y + Y_0 + \epsilon H$ (since the coefficients of E_i in $\mathbf{B}_{-}^{\mathrm{div}}(K_Y + Y_0 + \epsilon H)$ is also positive if $\operatorname{coeff}_{E_i} \mathbf{B}_{-}^{\mathrm{div}}(K_Y + Y_0) > 0$). Then, by Proposition 5.7, any MMP will contract those E_i.

Step 5. Prove fiberwise bimeromorphic. By the Cone theorem, those divisors contracted will be covered by rational curves. However, we assume that X_0^Y is not uniruled (thus, it is not contracted by the MMP). By Theorem 2.4 the generic fiber of $X \dashrightarrow Y^m$ is bimeromorphic, so one only needs to prove that the central fiber X_0 is bimeromorphic to Y_0^m. In fact, since the only component on Y_0^m is the strict transform of X_0^Y, X_0 is bimeromorphic to Y_0^m.

Step 6. Check the singularity assumptions. Note that the fibers Y_s of the family $h : Y \to \mathbb{D}$ is smooth away from Y_0 (by the semi-stable assumption) thus $(Y_s, \epsilon H_s)$ is terminal for $s \neq 0$ and $0 \leq \epsilon \ll 1$ (see [KM98, Corollary 2.35. (2)])

Since H_s is ample, by negativity lemma the MMP above will not contract H_s. Note that $(Y_s^{\mathrm{m}}, \epsilon H_s^{\mathrm{m}})$ is still terminal (by [KM98, Corollary 3.43]). Thus, Y_s^{m} also admits the terminal singularity (see [KM98, Corollary 2.35]). Since $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is DLT, it's also PLT thanks to the irreducible of Y_0^{m} ([KM98, Proposition 5.51]). And therefore Y_0^{m} is KLT by the easy direction of inversion of adjunction (see Theorem 5.3). $\square$

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General type Locus, Moishezon Locus and Projective Locus Summer 2025

Note 3 — 2025-07-12 (draft version)

Yi Li

1 Overview

The aim of this note is to study the distribution of general type locus, Moishezon locus, and projective locus (see Definition 2.1) on the irreducible base. The motivation of this topic comes from the following observation on the distribution of polarized (projective) K3 surfaces in the universal family of marked complex K3 surfaces.

Let $X \to D^{20}$ be a universal family of K3 surfaces. A smooth, compact surface is Moishezon iff it is projective. The projective fibers of $X \to D^{20}$ correspond to a countable union of hypersurfaces $H_{2g} \subset D^{20}$. As we can see from this example, the projective locus (which corresponds to projective K3 surfaces) is a countable union of the hypersurface in the moduli space D^{20}.

It is natural to ask how the locus of fibers that admits certain properties is distributed on the base. This note focuses on the distribution of the fibers that are projective, of general type and Moishezon on the base S. The major references of this note are [Kol22a], and [Kol22b].

Contents

2 The alternating property of the very big locus, general type locus

We first give the definitions for the very big locus, Moishezon locus, general type locus, and the projective locus.

Definition 2.1 (Very big locus, general type locus, Moishezon locus, [Kol22a, Definition 18])**.** Let $g : X \to S$ be a proper morphism of normal analytic spaces and L a line bundle on X. Set

1. $\mathrm{VB}_S(L) := \{s \in S : L_s \text{ is very big on } X_s\} \subset S$,

2. $\mathrm{GT}_S(X) := \{s \in S : X_s \text{ is of general type } \} \subset S$,
3. $\mathrm{MO}_S(X) := \{s \in S : X_s \text{ is Moishezon } \} \subset S$,
4. $\mathrm{PR}_S(X) := \{s \in S : X_s \text{ is projective } \} \subset S$.

Here very big means the place $s \in S$ that $X_s \dashrightarrow \mathrm{Proj}_S(g_* L_s)$ is bimeromorphic onto its closure of the image.

Definition 2.2 (Locus V that satisfies the alternating property over S)**.** Let $g : X \to S$ be a proper morphism of normal analytic spaces, we say the locus

$$V := \{s \in S \mid X_s \text{ admits property } P\},$$

satisfies *the alternating property over* S if $V \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S.

Remark 2.3. In general: (a) A subset which is not nowhere dense does not need to contain an open subset of S. e.g. $\mathbb{Q} \subset \mathbb{R}$ is not nowhere dense but it clearly contains no dense open subset of $\mathbb{R}$. (b) A subset that is not nowhere dense does not need to contain a dense subset of S as well, e.g. the disc $\mathbb{D} = \{|z| < 1\} \subset \mathbb{C}$ is not nowhere dense, but it is also not dense in $\mathbb{C}$.

In the analytic Zariski topology. If S is irreducible, and $V \subset S$ is not nowhere dense, then V is dense in S. To see this, by definition, $\bar{V}$ contains a non-empty Zariski open subset of S. Since S is irreducible, all the non-empty Zariski open subset is dense and therefore $\bar{V} = S$.

Note that the property that $V \subset S$ satisfies the alternating property over S does not care about the information on the special fibers. In other words, we have the following lemma.

Lemma 2.4. Assume that S is irreducible, if V satisfies the alternating property on some non-empty Zariski open subset $S' \subset S$, then V also satisfies the alternating property on S.

Proof. Since S is irreducible, the non-empty Zariski open subset $S' \subset S$ is Zariski dense in S. Then we have two cases:

Case 1. If V is nowhere dense in S', then V is also nowhere dense in S. By contradiction, if there exists some non-empty Zariski open subset W (of S) contained in V. Since S is irreducible, the intersection $W \cap S'$ is a non-empty Zariski open subset of S'. And therefore it contradicts to the nonwhere dense of V in S'.

Case 2. If V is dense in S' and we know that $S' \subset S$ is Zariski dense, then V is also dense in S.

□

Note that local system on irreducible complex variety is trivial.

Lemma 2.5. Local systems on an irreducible algebraic variety with the Zariski topology are trivial.

Proof. Since X is irreducible iff any non-empty intersection of the Zariski open subsets is non-empty. And by definition, for any point $x \in X$, there exists an open subset that the local system is constant

$$\mathscr{L}|_{U_i} = \underline{S_i},$$

. And any such $U_i \cap U_j \neq \emptyset$ so that

$$\mathcal{L}|_{U_i \cap U_j} = \underline{S_i}|_{U_i \cap U_j} = \underline{S}_j|_{U_i \cap U_j},$$

so that $\mathcal{L}$ is constant on $U_i \cup U_j$. By quasi-compactness, we know that the local system is automatically constant. $\square$

We first show that the very big locus satisfies the alternating property.

Proposition 2.6. Let $f : X \dashrightarrow Y/S$ be a proper morphism, between complex analytic varieties. Assume that the restriction on each fiber $f_s : X_S \dashrightarrow Y_s$ are bimeromorphic, can we prove that $f : X \to Y/S$ is bimeromorphic S-map?

Theorem 2.7 (Alternating property of very big locus, [Kol22a, Lemma 19])**.**
Let $g : X \to S$ be a proper morphism of normal irreducible analytic spaces and L a line bundle on X. Then $\mathrm{VB}_S(L) \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon.

Proof. We may assume that $g : X \to S$ is surjective (otherwise by properness of g, it will immediately in (1)). By Lemma 2.4, it is possible to pass to a non-empty Zariski open subset of S. Thus, we may assume that g is flat, g_*L is locally free and commutes with restriction to the fibers. We get a meromorphic map $\phi : X \dashrightarrow \mathbb{P}_S(g_*L)$. There is thus a smooth, bimeromorphic model $\pi : X' \to X$ such that $\phi \circ \pi : X' \to \mathbb{P}_S(g_*L)$ is a morphism. After replacing X by X' and again passing to an open subset of S, we may assume that g is flat, g_*L is locally free, commutes with restriction to fibers, and $\phi : X \to \mathbb{P}_S(g_*L)$ is a morphism.

Let $Y \subset \mathbb{P}_S(g_*L)$ denote its image and $W \subset X$ the Zariski closed set of points where $\pi : X \to Y$ is not smooth. Set $Y^\circ := Y \backslash \phi(W)$ and $X^\circ := X \backslash \phi^{-1}(\phi(W))$. The restriction $\phi^\circ : X^\circ \to Y^\circ$ is then smooth and proper. We divide the discussion into two cases:

Case 1. If we assume that the set of points

$$E = \{y \in Y \mid \phi^{-1}(y) \text{ is single points}\} \subset Y,$$

is not dense in Y. We claim in this case the $\mathrm{VB}_S(L)$ is nowhere dense in S. For otherwise, it will imply that $\mathrm{VS}_S(L)$ is dense in S. And, so that for dense set of fibers $\{X_s\}_{s \in \mathrm{VB}_S(L)} \subset X$, the restriction of the relative Kodaira map are bimeromorphic onto its image. In particular, the $E_s = E \cap X_s \subset X_s$ is dense in X_s. We claim that this will imply that

$$E = \bigcup_{s \in S} E_s,$$

is dense in Y which will give the contradiction. This is because

$$\bigcup_{s \in \mathrm{VB}_S(L)} Y_s = \bigcup_{s \in \mathrm{VB}_S(L)} \mathrm{cl}_{X_s}(E_s) \subset \bigcup_{s \in \mathrm{VB}_S(L)} \mathrm{cl}_X(E_s) \subset \mathrm{cl}_X(\bigcup_{s \in \mathrm{VB}_S(L)} E_s)$$

Case 2. E is dense set in Y, thus it's also for a dense set in Y°. Since ϕ^o is proper and smooth, ϕ^o is a finite étale morphism of degree 1, thus it is an isomorphism.

Thus, ϕ is bimeromorphic on every irreducible fiber that has a non-empty intersection with X°. That is, if we denote $D := \{s \in S \mid X_s \cap X^o \neq \emptyset\} \bigcap \{s \in S \mid X_s \text{ is irreducible}\}$ with $g(X^o) = \{s \in S \mid X_s \cap X^o \neq \emptyset\}$, then

$$D \subset \mathrm{VB}_S(L),$$

(1) Note that irreducible of the fiber X_s is needed, if $X_s \cap X^o \neq \emptyset$ and X_s is irreducible, then $X^o \cap X_s \subset X_s$ is an non-empty Zariski open subset of X_s, which is dense on the fiber X_s. Note again since both X and S are irreducible, the generic fibers of g are irreducible, see [GW20, Exercise 6.15]. Thus adding this constraint will not change the result),
(2) Note that the very big locus is not directly defined by the restriction of $X \to \mathbb{P}_S(g_*L)$ on the fibers. Instead, it's defined by the Kodaira map $X_s \to \mathbb{P}(H^0(X_s, L_s)$. Since we assume that g_*L commutes with restriction on the fiber, these two Kodaira maps coincide.

Recall that a morphism between analytic varieties will send a dense subset to a dense subset in its image. And g is flat (by assumption at the beginning), so that g is open. Thus g will send a Zariski dense open subset to a Zariski dense open subset. Thus D is a non-empty dense Zariski open subset contained in the $\mathrm{VB}_S(L)$.

Finally, we need to show that in this case $g : X \to S$ is a Moishezon morphism, i.e. the relative Kodaira map over S induced by L is bimeromorphic onto its image. Since $\phi^o : X^o \to Y^o$ is an isomorphism for $X^o \subset X$ an non-empty dense open subset, the result follows. □

As a direct consequence (combined with birational boundedness result of Hacon-Mckernan [HM06]) we see the general type locus also admits the alternating property.

Theorem 2.8 (The alternating property of the general type locus, [Kol22a, Corollary 20])**.**
Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then the general type locus

$$\mathrm{GT}_S(X) = \{s \in S \mid X_s \text{ is of general type}\},$$

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon

Proof. Using resolution of singularities, we may assume that X is smooth. By passing to an open subset of S, we may also assume that S and g are smooth. By [HM06] there is an m (depending only on $\dim X_s$) such that $|mK_{X_s}|$ is very big whenever X_s is of general type. Thus, Theorem 2.7 applies to $L = mK_X$. □

3 The alternating property of the Moishezon locus

In this section, we will prove that the Moishezon locus also admits certain alternating property. Before proving Theorem 3.6. Let us first introduce the following result, by [RT22].

Definition 3.1. Let $\mathcal{X}$ be a complex manifold, $\Delta \subseteq \mathbb{C}$ the unit disk and $f : \mathcal{X} \to \Delta$ a flat family, smooth over the punctured disk Δ^*. We say that f is a one-parameter degeneration.

Theorem 3.2 (Moishezon morphism criterion, [RT22, Proposition 3.15])**.** Let $\pi : \mathcal{X} \to \Delta$ be a smooth morphism.
(1) Assume that there exists an uncountable subset B of Δ such that for each $t \in B$, the fiber X_t admits a line bundle L_t with the property that $c_1(L_t)$ comes from the restriction to X_t of some cohomology class in $H^2(\mathcal{X}, \mathbb{Z})$.
(2) Assume further that the Hodge number $h^{0,2}(X_t) := h^2(X_t, \mathcal{O}_{X_t})$ is independent of $t \in \Delta$ (the original theorem requires only Hodge (0,1) deformation invariance).

Then there exists a global line bundle L over $\mathcal{X}$ such that $c_1(L|_{X_s}) = c_1(L_s)$ for any s in some uncountable subset of B.

Proof. Apply the sheaf exponential exact sequence so that

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}^* \to 0.$$

We claim that

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong H^0(\Delta, R^2\pi_*\mathcal{O}_X),\ H^2(X_s, \mathcal{O}_{X_s}) \cong R^2\pi_*\mathcal{O}_{\mathcal{X}}(s).$$

Indeed:

(1) By Cartan B. we have

$$H^p(S, R^q\pi_*\mathcal{O}_X) = 0,\ p > 0, q \geq 0,$$

and the Leray spectral sequence argument implies the first one,
(2) Since we assume the cohomological dimension $h^{0,2}$ is constant, by Grauert base change theorem, the second one follows.

Thus, we have the following commutative diagram.

$$\begin{tikzcd}
 & & & H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \arrow[d, "\cong"] & \\
\arrow[r] & H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r, "e_2"] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H^1(X_s, \mathcal{O}_{X_s}^*) \arrow[r] & H^2(X_s, \mathbb{Z}) \arrow[r, "e_2"] & H^2(X_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d, "\cong"] & {} \\
 & & & R^2\pi_*\mathcal{O}_{\mathcal{X}}(s) &
\end{tikzcd}$$

Where we have the evaluation $\mathrm{ev}_s : H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \to R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ in the diagram above.

Let $L_s \in \mathrm{Pic}(X_s)$ such that $c_1(L_s) \in H^2(X_s, \mathbb{Z})$. By simply connectedness of Δ, $c_1(L_s) \in H^2(X_s, \mathbb{Z})$ will lift to $c \in H^2(\mathcal{X}, \mathbb{Z})$. If we can prove the vanishing of $e_2(c) \in H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ then by the exactness of the sequence we can find some global line bundle $L \in \mathrm{Pic}(\mathcal{X})$.

Observe that the cohomology group $H^2(X_s, \mathbb{Z}) \cong H^2(\mathcal{X}, \mathbb{Z})$ is $\mathbb{Z}$ coefficient, so that it has only countable many elements. Given uncountable many L_t, it must have some $c \in H^2(\mathcal{X}, \mathbb{Z})$ such that uncountable subset of t satisfies $c_1(L_t) = c$.

Since this $c \in H^2(X_s, \mathbb{Z})$ comes from $L_s \in \mathrm{Pic}(X_s)$, we have $e_2(c) = 0 \in R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ and thus if we lift it to $c \in H^2(\mathcal{X}, \mathbb{Z})$ the global section $e_2(c) \in H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$ will vanish on uncountable many points. Thus by the identity principle (since $R^2\pi_*\mathcal{O}_{\mathcal{X}}$ is locally free (this step is not due to torsion freenss, we need Hodge number condition to get torsion freeness) the vanishing locus of $e_2(c)$ is a subvariety), we have $e_2(c) = 0 \in H^2(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$. Therefore, there exists a global line bundle $L \in \mathrm{Pic}(\mathcal{X})$ with the restriction $c_1(L|_{X_s}) = c_1(L_s)$. Finally, by the Proposition 3.3 deformation density of Iitaka-Kodaira dimension, we conclude that L is indeed a global big line bundle. □

The following proposition is used in the proof of Theorem 3.1.

Proposition 3.3 (Deformation density of Iitaka-Kodaira dimension, [LS77, Theorem 3.4])**.** Let $\pi : \mathcal{X} \to Y$ be a flat family from a complex manifold over a one-dimensional connected complex manifold Y with possibly reducible fibers. If there exists a holomorphic line bundle L on $\mathcal{X}$ such that the Kodaira-Iitaka dimension $\kappa(L_t) = \kappa$ for each t in an uncountable set B of Y, then any fiber X_t in π has at least one irreducible component C_t with $\kappa\left(L|_{C_t}\right) \geq \kappa$.

In particular, if any fiber X_t for $t \in Y$ is irreducible, then $\kappa(L_t) \geq \kappa$.

We next add some supplementary materials about the sheaf exponential sequence and relative Picard functor.

Lemma 3.4 ([Har77, p. 466])**.** Let X be a reduced complex analytic space, then the following sheaf exponential sequence is exact.

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

The following proposition relative the relative Picard functor

Proposition 3.5 (Picard-Brauer exact sequence)**.** Let $X \to T$ be a proper surjective morphism between complex varieties.

$$\begin{aligned} 0 \to \mathrm{H}^1\left(T, f_{T*}\mathcal{O}_{X_T}^*\right) \to \mathrm{H}^1\left(X_T, \mathcal{O}_{X_T}^*\right) \to \mathrm{H}^0\left(T, \mathrm{R}^1 f_{T*}\mathcal{O}_{X_T}^*\right) \\ \to \mathrm{H}^2\left(T, f_{T*}\mathcal{O}_{X_T}^*\right) \to \mathrm{H}^2\left(X_T, \mathcal{O}_{X_T}^*\right), \end{aligned}$$

we call $H^2(T, f_{T*}\mathcal{O}_{X_T}^*) = H^2(T, \mathcal{O}_T^*)$ the Brauer group. Note that a global section $H^0(T, R^1(f_T)_*\mathcal{O}_{X_T}^*)$ that comes from $H^0(X_T, R^2(f_T)_*\mathbb{Z})$ will automatically vanishing in the Brauer group since we have the factorization $H^2(T, \mathcal{O}_{X_T}^*)$ as

$$H^0(T, R^1(f_T)_*\mathcal{O}_{X_T}^*) \to H^0(T, R^2(f_T)_*\mathbb{Z}_{X_T}) \to H^0(T, R^2(f_*)).$$

If $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, then $\mathrm{H}^1(T, \mathcal{O}_T^*) \xrightarrow{\sim} \mathrm{H}^1\left(T, f_{T*}\mathcal{O}_{X_T}^*\right)$. Hence we have the following Picard-Brauer exact sequence,

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)(\mathrm{zar})}(T).$$

We now turn to the proof of the alternating property of the Moishezon locus.

Theorem 3.6 ([Kol22a, Theorem 21])**.** Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces. Then $\mathrm{MO}_S(X) \subset S$ is

(1) either contained in a countable union $\cup_i Z_i$, where $Z_i \subsetneq S$ are Zariski closed,
(2) or $\mathrm{MO}_S(X)$ contains a dense, open subset of S.

Furthermore, if $R^2 g_* \mathcal{O}_X$ is torsion free then (2) can be replaced by
(3) $\mathrm{MO}_S(X) = S$ and g is locally Moishezon.

Remark 3.7. The condition (1) is slightly different from the nowhere dense condition compared with Lemma 2.7 and Theorem 2.8. Indeed the countable union of nowhere dense subset needs not to be nowhere dense (e.g. $\mathbb{Q}$ as countable union of nowhere dense subset is no longer nowhere dense). As we will see in the proof, this replacement is necessary. Another difference compared with Lemma 2.7 and Proposition 2.8 is here we assume the morphism is smooth.

Remark 3.8. Compared with the proof of [RT22], Kollár's proof does not require the base to be the unit disc Δ. Consequently, the direct image $R^2 g_* \mathcal{O}_X$ is only torsion free, which does not need to be locally free.

Proof. Assume first that $R^2 g_* \mathcal{O}_X$ is torsion free. The sheaf exponential sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1.$$

gives

$$R^1 g_* \mathcal{O}_X^* \to R^2 g_* \mathbb{Z}_X \xrightarrow{e_2} R^2 g_* \mathcal{O}_X.$$

We may pass to the universal cover of S. Note that the local system on the simply connected space is constant, thus $R^2 g_* \mathbb{Z}_X \otimes \mathcal{O}_S$ is a trivial bundle.

Let $\{\ell_i\}$ be those global sections of $R^2 g_* \mathbb{Z}_X$ such that $e_2(\ell_i) \in H^0(S,\ R^2 g_* \mathcal{O}_X)$ is identically 0, and $\left\{\ell'_j\right\}$ the other global sections (those $\{\ell_i, \ell'_i\}$ are countable since we consider the $\mathbb{Z}$-coefficient local system). The ℓ_i then lift back to the global sections of $R^1 g_* \mathcal{O}_X^*$. Hence to line bundles L_i on X. We then divide the problem into two cases:

Case 1. If there is an L_i such that $\mathrm{VB}_S(L_i)$ contains a dense open subset of S, then $X \to S$ is Moishezon (by Proposition 2.7) and we are done.

Case 2. If any such line bundle L_i has nowhere dense very big locus $\mathrm{VB}_S(L_i)$. We claim

$$\mathrm{MO}_S(X) \subset \cup_i \mathrm{VB}_S(L_i) \bigcup \cup_j \left(e_2\left(\ell'_j\right) = 0\right).$$

If $s \in \mathrm{MO}_S(X)$, and $s \notin \cup_j \left(e_2\left(\ell'_j\right) = 0\right)$. We claim in this case every line bundle on X_s is numerically equivalent to some $L_i|_{X_s}$. For otherwise, there exist a line bundle L_s on X_s, with $c_1(L_s)$ lift to some ℓ'_j. Since the diagram below commute, which means that $\mathrm{ev}_s(e_2(\ell'_j)) = e_2(c_1(L_s)) = 0$ must vanish, contradict to the $s \notin \left(e_2\left(\ell'_j\right) = 0\right)$.

$$\begin{tikzcd}
H^0(S, R^2 g_* \mathbb{Z}) = H^2(S, \mathbb{Z}) \arrow[r, "e_2"] \arrow[d, "\simeq"'] & H^0(X, R^2 g_* \mathcal{O}_X) \arrow[d, "\mathrm{ev}_s"] \\
H^2(X_s, \mathbb{Z}) \arrow[r, "e_2"'] & R^2 g_* \mathcal{O}_X(s) \simeq H^0(X_s, \mathcal{O}_{X_s})
\end{tikzcd}$$

(Note that the isomorphism $H^2(X_s, \mathcal{O}_s) \simeq R^2 g_* \mathcal{O}_X(s)$ at the point $s \in \mathrm{MO}_S(X)$ since locally free of $R^2 g_* \mathcal{O}_X$ in neighborhood of $s \in S$ using Proposition 3.9 and the Hodge decomposition we proved in the first time).

Thus X_s has a big line bundle (as $s \in \mathrm{MO}_S(X)$) $\Leftrightarrow L_i|_{X_s}$ is big for some $i \Leftrightarrow L_i|_{X_s}$ is very big for some i (and therefore $s \in \cup_i \mathrm{VB}_S(L_i)$). This completes the case when $R^2 g_* \mathcal{O}_X$ is torsion free. □

We next show that fiberwise Moishezon morphism is locally Moishezon if the morphism is smooth. Before proving the result, let us give a locally free criterion of direct image when the fibers satisfy the Du Bois property.

Theorem 3.9 (Locally freeness criterion for $R^i f_* \mathcal{O}_X$, [Kol22a, Theorem 24])**.** Let $f : X \to S$ be a smooth, proper morphism of analytic spaces. Assume that $H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s})$ is surjective for every i for some $s \in S$. Then $R^i g_* \mathcal{O}_X$ is locally free in a neighborhood of s for every i.

Proof. We begin our proof by noticing by the direct image theorem it's enough to show the surjectivity of the base change morphism

$$\phi_s^i : R^i f_* \mathcal{O}_X(s) \to H^i(X_s, \mathcal{O}_{X_s}),$$

for every i. Indeed the base change theorem shows that the surjectivity of the base change morphisms ϕ_s^i and ϕ_s^{i-1} implies the locally freeness of the direct image $R^i f_*(\mathcal{O}_X)$ (see Hartshorne Corollary 12.9).

Next by the Theorem on Formal Functions, it is enough to prove this when S is replaced by any Artinian local scheme S_n, whose closed point is s.

By Cartan B easy to see the vanishing of $H^p(S_n, R^i f_* \mathcal{O}_X) = 0, \ \forall q, \forall i > 0$ then by the Leray spectral sequence argument we get

$$H^0\left(S_n, R^i f_* \mathcal{O}_X\right) = H^i\left(X_n, \mathcal{O}_{X_n}\right), \quad \text{for } i \geq 0.$$

On the local Artinian base with the closed point s, we have the following equality

$$R^i f_* \mathcal{O}_X(s) = H^0(S_n, R^i f_* \mathcal{O}_X) = H^i(X_n, \mathcal{O}_{X_n}).$$

The base change morphism thus becomes

$$\psi^i : H^i(X_n, \mathcal{O}_{X_n}) \to H^i(X_s, \mathcal{O}_{X_s}).$$

Let $\mathbb{C}_{X_n}$ (resp. $\mathbb{C}_{X_s}$) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n : \mathbb{C}_{X_n} \to \mathcal{O}_{X_n}$ (resp. $j_s : \mathbb{C}_{X_s} \to \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$\begin{tikzcd}
H^i(X_n, \mathbb{C}_{X_n}) \arrow[r, "\alpha^j"] \arrow[d, "j_n'"'] & H^i(X_s, \mathbb{C}_{X_s}) \arrow[d, "j_s'"] \\
H^i(X_n, \mathcal{O}_{X_n}) \arrow[r, "\psi^j"'] & H^i(X_s, \mathcal{O}_{X_s})
\end{tikzcd}$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism, and j_s^i is surjective by assumption. Thus ψ^i is also surjective. □

Using this we can prove that the smooth fiberwise Moishezon morphism is locally Moishezon morphism.

Theorem 3.10 (Fiberwise Moishezon smooth morphism is locally Moishezon , [Kol22a, Corollary 22]). Let $g : X \to S$ be a smooth, proper morphism of normal and irreducible analytic spaces whose fibers are Moishezon. Then g is locally Moishezon.

Proof. Since we proved (in the first time) the Moishezon manifolds admit strong Hodge decomposition, thus

$$H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s}),$$

is surjective for every $i \geq 0$. The result then follows directly by Theorem 3.9. □

4 The alternating property of the projective locus

In the last section, we will finish the proof of the alternating property about the projective locus. The following Thom Whitney stratification theorem is useful in the proof.

Proposition 4.1 (Thom Whitney stratification theorem, [Kol22b, Lemma 15]). Let $f : X \to S$ be a proper morphism of complex analytic spaces. There exist finite Whitney stratifications $\mathcal{X}$ of X and $\mathcal{S} = \{S_l\}_{l \leq d}$ of S by locally closed subsets S_l of dimension l, with $d = \dim S$, such that for each connected component S (a stratum) of S_l. The following condition holds.

(a) $f^{-1}S$ is a topological fibre bundle over S, union of connected components of strata of $\mathcal{X}$, each mapped submersively to S,

(b) For all $v \in S$, there exist an open neighborhood $U(v)$ in S and a stratum preserving homeomorphism $h : f^{-1}(U) \simeq f^{-1}(v) \times U$ s.t. $f_{|U} = p_U \circ h$ where p_U is the projection on U.

In particular, there is a dense, Zariski open subset $S^\circ \subset S$ such that $g^\circ : X^\circ \to S^\circ$ is a topologically locally trivial fiber bundle. Moreover, If $S = \Delta$, if we shrink the disc then $f : X^* \to \Delta^*$ is topologically fiber bundle.

Under this assumption, we can prove the local system $R^i g_* \mathbb{Z}_X$ is constructible in the analytic Zariski topology for a proper morphism between complex analytic spaces.

Corollary 4.2 ([Kol22b, Corollary 16]). Let $g : X \to S$ be a proper morphism of complex analytic spaces. Then the sheaves $R^i g_* \mathbb{Z}_X$ are constructible in the analytic Zariski topology.

When consider the global section of a local system, the following result is helpful.

Lemma 4.3. Let $\mathscr{L}$ be a local system on a complex manifold S, the global section

$$H^0(S, \mathscr{L}) = L^\rho := \{a \in L |\ \rho(\alpha)(a) = a, \forall \alpha \in \pi_1(S, v)\},$$

where L is the fiber of the local system on the reference point $v \in S$. And $\rho : \pi_1(S, v) \to GL(L)$ be the monodromy action. In particular if the base S is simply connected, then $H^0(S, \mathscr{L}) = L$.

Proposition 4.4 (The alternating property of projective locus, [Kol22b, Proposition 17])**.** Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that
(1) either X is locally projective over S°,
(2) or $\mathrm{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

If g is bimeromorphic to projective morphism, then X is projective over S^o.

Remark 4.5. The locally projective condition is necessary in some situations (that is $g : X \to S$ may not be projective over S^o). Question to be done: where do we use the "locally" in the proof? Euclidean topology or Zariski topology?

Proof. If we restrict our attention to the main strata S^o of the Whitney stratification, the direct image $R^2 g_* \mathbb{Z}_X$ is locally constant. And further restricting on some Zariski open subset, we can also assume that $R^2 g_* \mathcal{O}_X$ is locally free. By passing to the universal cover, we may also assume that $R^2 g_* \mathbb{Z}_X$ is a constant sheaf. Now consider the sheaf exponential sequence

$$R^1 g_* \mathcal{O}_X^* \to R^2 g_* \mathbb{Z}_X \xrightarrow{\partial} R^2 g_* \mathcal{O}_X.$$

Let Θ be a global section of $R^2 g_* \mathbb{Z}_X$. By Lemma 4.3, we know that $\Theta \in H^2(X, \mathbb{Z}) = H^0(S, R^2 g_* \mathbb{Z}_X)$. We decompose the cohomology into two disjoint parts,

$$H^2(X, \mathbb{Z}) = V_1 \sqcup V_2,$$

where

$$V_1 = \{\Theta \in H^2(X, \mathbb{Z}) \mid \partial\Theta \equiv 0\},\ V_2 = \{\Theta \in H^2(X, \mathbb{Z}) \mid \partial\Theta \not\equiv 0\}.$$

Since we assume that $R^2 g_* \mathcal{O}_X$ is a vector bundle, the vanishing locus is a Zariski closed nowhere dense subset we denote $H_\Theta = V(\Theta)$ for $\Theta \in V_2$.

Case 1. Given a point $s \in PR_S(X)$, there exists some ample line bundle L_s on X_s, and thus under the exact sequence

$$\mathrm{Pic}(X_s) \to H^2(X_s, \mathbb{Z}) \xrightarrow{\partial} H^2(X_s, \mathcal{O}_{X_s}),$$

$c_1(L_s) = \Theta_{L_s}$ maps to some zero element $\partial(\Theta_{L_s}) = 0$. Since

$$\mathrm{res}_s : H^2(X, \mathbb{Z}) \xrightarrow{\simeq} H^2(X_s, \mathbb{Z}),$$

one can lift the class $\Theta_{L_s} \in H^2(X_s, \mathbb{Z})$ to a class $\Theta \in H^2(X, \mathbb{Z})$.

If $\partial\Theta$ is identically zero, then it lifts to a line bundle $L \in \mathrm{Pic}(X)$, such that $L|_{X_s} = L_s$, which is ample and therefore by Grothendieck's ampleness theorem. We know that the morphism is locally projective around $s \in \mathrm{PR}_S(X)$.

Case 2. Assume that for all points $s \in \mathrm{PR}_S(X)$, all $\partial\Theta$ is not identically zero, then $\partial\Theta = 0$ defines a Zariski closed, nowhere dense subset $H_\Theta \subset S$. In this case, we know that

$$\Theta \in V_2,$$

and by the commutative diagram,

we know that $s \in V(\Theta)$. And thus

$$PR_S(X) \subset \cup_{\Theta \in V_2} V(\Theta).$$

$\square$

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Projectivity Criterira and Projective Stratification Summer 2025

Note 4 — 2025-07-16 (draft version)

Yi Li

1 Overview

The aim of this note is twofold.

(1) We summarize several projectivity criteria for Moishezon varieties. These include the singular version of Kodaira's projectivity criterion, the Nakai–Moishezon criterion, Seshadri's criterion, Kleiman's ampleness criterion, and a projectivity criterion for Moishezon morphisms as developed in [CH24].

(2) We discuss the projective stratification theorem. The ultimate goal is to complete the proof of the following result.

Theorem 1.1 ([Kol22, Theorem 2]). Let $g : X \to S$ be a proper Moishezon morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that X_0 is projective for some $0 \in S$, and the fibers X_s have rational singularities for $s \in S^*$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each $g|_{X_i} : X_i := g^{-1}(S_i) \to S_i$ is projective.

Contents

2 Projectivity critera

In this section, we summarize some projectivity criteria related to Moishezon varieties.

2.1 Kodaira's projectivity criterion

Proposition 2.1. Let X be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$, then X is projective.

Proof. Take the resolution $\nu : X' \to X$, where X' is a Kähler manifold. Since X has rational singularity, $R^i\nu_*\mathcal{O}_{X'} = 0$ for $i > 0$. Thus, by the Leray spectral sequence argument, $H^2(X, \mathcal{O}_X) = H^2(X', \mathcal{O}_{X'}) = 0$ and therefore by Kodaira's projectivity criterion for smooth manifolds, X' is projective. And therefore X is a Kähler Moishezon variety with rational singularity. By the result we proved in the first time, X is a projective variety. □

2.2 Nakai-Moishezon ampleness critera

Proposition 2.2 ([Kol90, Theorem 3.11])**.** Let X be a proper Moishezon space over $\mathbb{C}$ and let H be a line bundle on X. Then H is ample on X if an only if for every irreducible closed subspace $Z \subset X$, the intersection product $H^{\dim(Z)} \cdot Z$ is positive.

2.3 Seshadri criterion line bundle version

Seshadri constant was first introduced by Demailly in the early 90s, when he studied Fujita's conjecture.

Conjecture 2.3. Let X be a smooth projective variety of dimension n, with L being ample. Then

(a) $K_X + (n+1)L$ is global generated,

(b) $K_X + (n+2)L$ is very ample.

Definition 2.4. Given a proper analytic space X and a line bundle L, the *Seshari constant* is defined to be

$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\operatorname{mult}_x C}.$$

He tried to reduce Fujita's conjecture to the bound control of the Seshadri constant.

Theorem 2.5 ([Dem92])**.** Let X be a smooth projective variety of dimension n with L being ample. Then the following hold.
(a) If $\epsilon(L, x) > \frac{n}{n+1}$ then $K_X + (n+1)L$ is global generated,

(b) If $\epsilon(L, x) > \frac{2n}{n+2}$ then $K_X + (n+2)L$ is very ample.

For the readers who want to know more about this, please refer to [Dem92].

Proposition 2.6 ([Kol22])**.** Let X be a proper Moishezon space, and D a divisor on X (the same also true for $\mathbb{Q}$, $\mathbb{R}$ dvisor). Then D is ample if and only if there exists a positive number $\varepsilon > 0$ such that

$$\frac{(D \cdot C)}{\text{mult}\,_x C} \geq \varepsilon,$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x.

2.4 Seshadri criterion cohomology class version

Lemma 2.7. Let X be a normal compact Moishezon variety. Then the canonical map

$$\Phi : N^1(X) \to N_1(X)^\vee, \quad [D] \mapsto \lambda_D$$

is an isomorphism. Here we define

$$\lambda_D : N_1(X) \to \mathbb{R}, \quad [T] \mapsto T \cdot D.$$

Remark 2.8. For Fujiki varieties with rational singularity the result is also true:

Let X be a normal compact Fujiki variety with rational singularity. Then the canonical map

$$\Phi : N^1(X) \to N_1(X)^\vee, \quad \omega \mapsto \lambda_\omega$$

is an isomorphism. Here we define

$$\lambda_\omega : N_1(X) \to \mathbb{R}, \quad [T] \mapsto T(\omega).$$

Here

$$N^1(X) := H^{1,1}_{\mathrm{BC}}(X),$$

and $N_1(X)$ to be the vector space of real closed currents of bidimension $(1,1)$ modulo the following equivalence relation: $T_1 \equiv T_2$ if and only if

$$T_1(\eta) = T_2(\eta),$$

for all real closed $(1,1)$-forms η with local potentials.

Proposition 2.9 ([Kol22])**.** Let X be a proper Moishezon space over $\mathbb{C}$ with rational singularities. Then X is projective iff there is a cohomology class $\Theta \in H^2(X, \mathbb{Q})$ and an $\epsilon > 0$ such that

$$\Theta \cap [C] \geq \epsilon \cdot \text{mult}_p\, C$$

for every integral curve $C \subset X$ and every $p \in C$.

Proof. Note that the cup product induce a $\mathbb{Q}$-bilinear form

$$(-) \cap (-) : H^2(X, \mathbb{Q}) \times H_2(X, \mathbb{Q}) \to \mathbb{Q},$$

which will induce a $\mathbb{Q}$-linear functional on $H_2(X, \mathbb{Q})$. If $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X, \mathbb{Q})$, then we can view $C \mapsto \Theta \cap [C]$ as a $\mathbb{Q}$-linear map

$$\Theta \cap (-) : N_1(X, \mathbb{Q}) \to \mathbb{Q}.$$

By the previous lemma, $\Theta \cap (-)$ lies in the dual space $N^1(X, \mathbb{Q})$. And line bundles span the dual space of $N_1(X, \mathbb{Q})$. So there is a line bundle L on X and an $m > 0$ such that $\deg(L|_C) = m \cdot \Theta \cap [C]$ for every integral curve $C \subset X$. Thus

$$\deg(L|_C) = m \cdot \Theta \cap [C] \geq m\epsilon \cdot \operatorname{mult}_p C,$$

for every integral curve $C \subset X$ and every $p \in C$. Then L is ample by the line bundle version Seshadri criterion. Therefore X is projective.

Note $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$ if X has 1-rational singularities has been discussed in the first note. □

2.5 Klieman's ampleness criterion for Moishezon spaces

Proposition 2.10 ([VP21]). Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities and that L is a Cartier divisor on Y. Then L is ample if and only if L has positive degree on every irreducible curve on Y and L induces a strictly positive function on $\overline{\mathrm{NE}}(Y)$.

Remark 2.11. It remains open if the result is still true without the $\mathbb{Q}$-factorial KLT assumption.

Proof. The proof require the study of rational curves on Moishezon spaces, we will prove it in the next note. □

3 Approximation of the Chow-Barlet 1-cycle space

In this section, we will introduce the main technical tool: Chow-Barlet cycle space. We will proved that one can approximate the Chow-Barlet 1-cycle space using countable many families of marked curves, which is crucial for the proof of result Theorem 6.1.

Definition 3.1 (Chow functor with m-marked points, [Kol96, Definition I.3.20]). Let X be an analytic space over S. Let

$$\mathrm{Chow}_m(X/S)(Z) = \left\{ \begin{array}{c} \text{Well defined families of nonnegative,} \\ \text{proper, algebraic cycles } \mathcal{C} \text{ of } X \times_S Z/Z, \\ s_1, \dots, s_m : Z \to X, s_i(z) \in \mathcal{C}_z \text{ for all } z \in Z \end{array} \right\}.$$

We call the data in the bracket the *Chow data with m-marked points*. We say C is a *pointed curve* if it is a 1-cycle that has one marked point. And we denote the Barlet-Chow 1-cycle space with 1-marked point $\mathrm{Chow}^1_1(X/S)$.

Lemma 3.2 (Representative of the Chow functor with marked points). Let $X \to S$ be a proper morphism between complex analytic spaces. The relative Chow functor with m-marked points is representable by a complex analytic space $\mathrm{Chow}_m(X/S)$.

Proof. Since the proof does not appear in the standard references, for the completeness we add a proof here. We claim that Chow functor with marked points is actually represented by a closed subspace of the original Chow-Barlet cycle space (we call this closed subspace incident complex subspace). Let

$$\mathcal{U} \to \mathrm{Chow}(X/S),$$

be the universal family of the Barlet-Chow cycle space (with $\mathcal{U} \subset X \times_S \mathrm{Chow}(X/S)$ as closed complex subspace). We then define the m-fold fiber product to be $X^{(m)} = \underbrace{X \times_S X \times_S \ldots \times_S X}_{m\text{-times}}$.

Let $P = \mathrm{Chow}(X/S) \times_S X^{(m)}$, the incident complex subspace is defined to be

$$\mathrm{Chow}_m(X/S) = I = \{(s, x_1, \ldots, x_m) \in P \mid x_i \in \mathcal{U}_s, \text{ for all } i\}.$$

We claim that $I \subset P$ is a closed complex subspace. Indeed, we have the natural projective

$$p_i : P \to \mathrm{Chow}(X/S) \times_S X, \quad (c, x_1, \ldots, x_m) \mapsto (c, x_i),$$

and easy to check that the incidence variety can be represented as

$$I = \bigcap_{i=1}^{m} p_i^{-1}(\mathcal{U}),$$

since $\mathcal{U}$ is closed complex subspace in $X \times_S \mathrm{Chow}(X/S)$, and therefore as a finite intersection I is a closed complex subspace in P.

We then show that I is the representative of the Chow functor with marked points that is

$$\mathrm{Hom}_S(T, I) \simeq \mathrm{Chow}_m(X/S)(T).$$

To see this, we first show that given a S-morphism $T \to I/S$ it will induce a Chow data with marked points over S. Indeed, since $I \subset \mathrm{Chow}(X/S) \times_S X^{(m)}$, so that the first projection

$$\pi_1 : T \to I \to \mathrm{Chow}(X/S),$$

will induce a family over T via pull back. And the second projection

$$\sigma_i = \pi_{2,i} : T \to I \xrightarrow{q_i} X,$$

will defines the section we want. Conversely, given the Chow data $(\mathcal{Z}, \sigma_1, \ldots, \sigma_m)$ with marked point, it will induce a morphism. To see this, by the representative of the standard Chow functor, we know that there exists a morphism $\phi_{\mathcal{Z}} : T \to \mathrm{Chow}(X/S)$ such that $\mathcal{Z} \to T$ is the pull back family, with m-sections $\sigma_i : T \to X^{(m)}$. It is easy to check that the induced morphism actually maps into I,

$$\phi_{\mathcal{Z}} \times \sigma_i : T \to I \subset P.$$

□

The following upper semi-continuity result is needed in the proof.

Lemma 3.3 (upper semi-continuity of the multiplicities, [BM19, Proposition 4.3.10])**.** Let $(X_s)_{s \in S}$ be an analytic family of n-cycles of a complex space M. Then the function

$$S \times M \longrightarrow \mathbb{N}, (s, z) \mapsto \mathrm{mult}_z(X_s)$$

is upper semicontinuous in the Zariski topology of $S \times M$.

Proof. The proof of the lemma is a bit complicated and we omit it here. $\square$

Remark 3.4. In particular, let $f : X \to S$ be a proper flat morphism of relative dimension 1, assume that there is a holomorphic section $\sigma : S \to X$. Then the multiplicity

$$\mathrm{mult} : S \to \mathbb{Z}, \quad s \mapsto \mathrm{mult}_{\sigma(s)} X_s$$

is Zariski upper-semicontinuous.

Proof. Since the fibers $\{X_s\}$ clearly forms an anlytic family of cycles in X. Since the section map $\sigma : S \to X$ is holomorphic,

$$S \to S \times X \to \mathbb{N}, \quad s \mapsto (s, \sigma(s)) \mapsto \mathrm{mult}_{\sigma(s)} X_s,$$

is upper semi-continuous. $\square$

Theorem 3.5 (Approximation Chow-Barlet 1-cycle space, [Kol22])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces that is bimeromorphic to a projective morphism. Fix $m \in \mathbb{N}$. Then there are countably many diagrams of complex analytic spaces over S,

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, shift right, "w_i"'] & W_i \times_S X \\ W_i \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

indexed by $i \in I$, such that

(1) the $w_i : C_i \to W_i$ are proper, of pure relative dimension 1 and flat over a dense, Zariski open subset $W_i^\circ \subset W_i$,

(2) the fiber of w_i over any $p \in W_i^\circ$ has multiplicity m at $\sigma_i(p)$,

(3) the W_i are irreducible, the structure maps $\pi_i : W_i \to S$ are projective, and

(4) the fibers over all the W_i° give all irreducible curves that have multiplicity m at the marked point.

Proof. By assumption, there is a bimeromorphic morphism $r : Y \to X$ such that Y is projective over S.

$$\begin{tikzcd} Y \arrow[rr, "r"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

By Lemma, the Barlet-Chow cycle space of curves with marked points on Y/S exists (denote it $\mathrm{Chow}_1^1(Y/S)$) and its irreducible components W_i are projective over S. The universal family

$$\mathcal{C} \to \mathrm{Chow}_1^1(Y/S),$$

parameterize all pointed curves on Y in the fiber direction. Let W be any irreducible component of $\mathrm{Chow}_1^1(Y/S)$, We restrict the universal family on that component $C^Y \to W$.

We then map back the family of curves on Y:

$$\begin{tikzcd} C^Y \arrow[r, hook] \arrow[d, shift right, "w_Y"'] & W \times_S Y \\ W \arrow[u, shift right, "\sigma_Y"'] & \end{tikzcd}$$

to family of curves on X:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, shift right, "w"'] & W \times_S X \\ W \arrow[u, shift right, "\sigma"'] & \end{tikzcd}$$

(Note that the family $w : C \to W$ is no longer flat, as curves on the fibers can be contracted by $Y \to X$).

However, it's still proper flat over some dense Zariski open subset $W^o \subset W$. Since the family is flat over W^o, by Lemma 3.4, the multiplicity of a fiber C_w at the section s is an upper semi-continuous function on W°. For each $m \in \mathbb{N}$, let $W^m \subset W$ be the closure of the set of points $p \in W^\circ$ for which $\text{mult}_{\sigma(p)} C_p = m$. Since the restriction of a projective morphism over closed subvariety is still projective, $W^m \to S$ is a projective morphism.

We finally going back to the original Moishezon morphism $g : X \to S$. Let $X^\circ \subset X$ be the largest open set over which $r : Y \to X$ is an isomorphism. The above procedure gives all irreducible pointed curves that have nonempty intersection with X°. Equivalently, all curves with a marked point that are not contained in $X \backslash X^\circ$. We can now use dimension induction (Note that by the result we proved in the first time the restriction $X \setminus X^0 \to S$ is a Moishezon morphism, so that we can repeat the same argument). And we can get countably many families of pointed curves that approximate the Chow-Barlet 1-cycle space with 1-marked point. □

4 Projectivity of very general fibers

We can now prove the following theorem, which is the key step in deducing the main result.

Theorem 4.1 (Projectivity of very general fibers, [Kol22, Proposition 14])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that
(1) X_0 is projective for some $0 \in S$,
(2) the fibers X_s have rational singularities for $s \in S^*$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Euclidean open neighborhood $0 \in U \subset S$ and countably many nowhere dense, closed, analytic subsets $\{H_j \subset U : j \in J\}$, such that X_s is projective for every $s \in U \backslash \cup_j H_j$.

Proof. First choose $0 \in U \subset S$ such that X_U retracts to X_0. Since X_0 is projective, it carries an ample line bundle L. Let $\Theta \in H^2(X_U, \mathbb{Q})$ be the pull-back of $c_1(L)$ to X_U. Note that Θ is a topological cohomology class that is usually not the Chern class of a holomorpic line bundle. Let (C_s, p_s) be any marked curve on the fiber X_s for $0 \neq s \in U$.

Using Theorem 3.5, we can find countable many families of pointed curves, with projective morphisms $\pi_i : W_i \to U$.

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, shift right, "w_i"'] & W_i \times_S X \\ W_i \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

Let $J \subset I$ be the index such that $H_i := \pi_i(W_i) \subset U$ for $i \in J$ is nowhere dense in U. Therefore, $\pi_i : W_i \to U$ for $i \in I \setminus J$ will dominant U. Since π_i is projective, in particular it implies $0 \in \pi_i(W_i)$ for $i \in I \setminus J$.

Let $s \in U \setminus \cup_{j \in J} H_j$, then by definition of J, there is an $i \in I \backslash J$, such that the following conditions hold.

(a) (C_s, p_s) is one of the fibers of w_i over W_i°,

(b) $\mathrm{mult}_{\sigma_i(p)} C_p = m$ for all $p \in W_i^\circ$, and

(c) $\pi_i : W_i \to U$ is projective and its image contains $0, s \in S$ (say $\pi_i(0) = 0, \pi_i(w) = s$)

Since W_i is irreducible, there exist a holomorphic curve $\tau : \Delta \to W_i$ connecting the point $0, w$ (with $\tau(0) = 0, \tau(1) = w$ and the radius of $r(\Delta)\ > 1$). We then pull the family back to the disc

$$w : \mathcal{C} \to \Delta,$$

with section $\sigma : \Delta \to \mathcal{C}$. Note that

$$\mathrm{mult}_{\sigma(t)} \mathcal{C}_t = \mathrm{mult}_{\sigma(1)} \mathcal{C}_1 = \mathrm{mult}_{\sigma_i(s)} C_{is} \text{ for all } t \in \Delta^*,$$

since $\tau(\Delta^*) \subset W_i^o$. On the other hand, by the Lemma 3.4, we have

$$\mathrm{mult}_{\sigma(0)} \mathcal{C}_0 \geq \mathrm{mult}_{\sigma(t)} \mathcal{C}_t = \mathrm{mult}_{\sigma_i(s)} (C_i)_s, \quad \text{for } t \in \Delta^*.$$

(Here the pull back family $\mathcal{C} \to \Delta$ is flat, since the base is a disc and a surjective holomorphic map from reduced irreducible space to a disc is automatically flat).

Since $\mathcal{C}_0$ is a 1-cycle on the projective X_0, and $\Theta_0 = \Theta|_{X_0}$ is the Chern class of an ample line bundle on X_0. Thus

$$\Theta \cap [\mathcal{C}_0] \geq \epsilon \cdot \mathrm{mult}_{\sigma(0)} \mathcal{C}_0.$$

by the easy direction of Theorem 2.9, where ϵ depends only on X_0 and Θ_0.

Since $\mathcal{C}_0$ and $\mathcal{C}_1$ lie in the same irreducible component of Chow-Barlet cycle space, they are algebraic equivalent. Thus the cup product with Θ remain the same. Putting these together gives that

$$\boxed{\Theta_s \cap [C_s] = \Theta \cap [\mathcal{C}_1] = \Theta \cap [\mathcal{C}_0] \geq \epsilon \cdot \mathrm{mult}_{p_0} \mathcal{C}_0 \geq \epsilon \cdot \mathrm{mult}_{p_s} C_s}$$

Thus X_s is projective by another direction of Theorem 2.9. □

5 From locally projective to global projective

Lemma 5.1 (Trivialization of the monodromy after finite base change)**.** Let X be a connected complex analytic variety. Let $\mathscr{L}$ be a local system with finite monodromy defined on X. Then there exisfts a finite covering $\pi : X' \to X$ such that the pull back local system $\pi^*\mathscr{L}$ becomes trivial.

Proof. Let

$$\rho : \pi_1(X, x_0) \to \mathrm{GL}_n(L),$$

be the monodromy representation, with L be the fiber of the local system at the reference point $x_0 \in X$. Since the monodromy of $\mathscr{L}$ is finite, so that

$$\ker \rho \subset \pi_1(X, x_0),$$

is a finite index normal subgroup. Thus by the Galois correspondence, we can find a finite cover

$$\pi : X' \to X,$$

such that the fundamental group $\pi_*(\pi_1(X', x_0')) = \ker \rho \subset \pi_1(X, x_0)$ with $\pi(x_0') = x_0$. On the other hand, we have the following base change diagram for the monodromy representation.

$$\begin{array}{ccc} \pi_1(X', x_0') & \xrightarrow{\pi_*} & \pi_1(X, x_0) \\ \rho' \downarrow & & \downarrow \rho \\ \mathrm{GL}(L) & \xrightarrow[=]{} & \mathrm{GL}(L) \end{array}$$

so that the monodromy of $\rho' : \pi_1(X', x_0') \to \mathrm{GL}(L)$ is clearly trivial. □

Lemma 5.2. Let $g : X \to Y/S$ be a proper contraction morphism defined over S. The induced pull back map on the Néron-Sever group and N^1 space

$$g^* : \mathrm{NS}(Y/S) \to \mathrm{NS}(X/S), \quad g^* : N^1(Y/S) \to N^1(X/S),$$

are injective.

Proposition 5.3. Assume that $g : X \to S$ be a proper Moishezon morphism of normal irreducible analytic spaces. Assume that there exists a dense Zariski open subset $S^o \subset S$ such that X is locally projective over S^o then it's actually global projective.

Proof. By passing to a Zariski open subset, we may assume that $R^2 g_* \mathcal{O}_X$ is locally free, and $R^2 g_* \mathbb{Z}_X$ is locally constant. Thus by Proposition **??** and Lemma 5.2, after finite base change the Neron-Sever local system becomes trivial local system, i.e. the locally defined ample line bundle

$$L_i \in \mathcal{NS}(X/S)(U_i) = \mathcal{NS}(X/S)(S),$$

defines a global line bundle. □

6 Kollár's projective stratification theorem

Now we can prove the main theorem of this note.

Theorem 6.1 (Projective Stratification, [Kol22, Theorem 2])**.** Let $g : X \to S$ be a proper Moishezon morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,

(2) the fibers X_s have rational singularities for $s \in S^*$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \text{ is projective.}$$

Proof. By Theorem 4.1, we know that $\mathrm{PR}_S(X)$ contains the complement of a countable union of Zariski closed, nowhere dense subsets. By the Baire category theorem, $\mathrm{PR}_S(X)$ is not contained in a countable union of closed, nowhere dense subsets. And by the alternating property of projective locus that we proved in the previous note, we are in the case that $g : X \to S^o$ is locally projective over a dense, Zariski open subset $S^\circ \subset S$.

Since the morphism is Moishezon, therefore by [Kol22, Complement 18], the morphism $g : X \to S$ is global projective over S^o. And we repeat the process on $S \setminus S^o$ gives the stratification of $g : X \to S$ into projective morphisms $g|_{X_i} : X_i = g^{-1}(S_i) \to S_i$. □

7 Claudon-Höring's projectivity criterion for Kähler morphisms

In this section, we introduce the following projectivity criterion for Kähler morphism.

Theorem 7.1 ([CH24, Theorem 3.1])**.** Let $f : X \to Y$ be a fibration between normal compact Kähler spaces. Assume that X has strongly $\mathbb{Q}$-factorial KLT singularities. Assume one of the following:
(1) The normal space Y has klt singularities and the natural map

$$f^* : H^0\left(Y, \Omega_Y^{[2]}\right) \longrightarrow H^0\left(X, \Omega_X^{[2]}\right)$$

is an isomorphism.
(2) The morphism f is Moishezon.

Then f is a projective morphism.

Proof. We will discuss this in the next note. □

Final words, Projectivity of moduli has been systematic studied by Kollár in the 1990's. For readers who want to know more about this direction, please refer to [Kol90].

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