Projectivity Criterira and Projective Stratification Summer 2025

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1 Overview

The aim of this note is twofold.

(1) We summarize several projectivity criteria for Moishezon varieties. These include the singular version of Kodaira's projectivity criterion, the Nakai–Moishezon criterion, Seshadri's criterion, Kleiman's ampleness criterion, and a projectivity criterion for Moishezon morphisms as developed in [CH24].

(2) We discuss the projective stratification theorem. The ultimate goal is to complete the proof of the following result.

Theorem 1.1 ([Kol22, Theorem 2])**.** Let $g : X \to S$ be a proper Moishezon morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that X_0 is projective for some $0 \in S$, and the fibers X_s have rational singularities for $s \in S^*$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each $g|_{X_i} : X_i := g^{-1}(S_i) \to S_i$ is projective.

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2 Projectivity critera

In this section, we summarize some projectivity criteria related to Moishezon varieties.

2.1 Kodaira's projectivity criterion

Proposition 2.1. Let X be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$, then X is projective.

Proof. Take the resolution $\nu : X' \to X$, where X' is a Kähler manifold. Since X has rational singularity, $R^i\nu_*\mathcal{O}_{X'} = 0$ for $i > 0$. Thus, by the Leray spectral sequence argument, $H^2(X, \mathcal{O}_X) = H^2(X', \mathcal{O}_{X'}) = 0$ and therefore by Kodaira's projectivity criterion for smooth manifolds, X' is projective. And therefore X is a Kähler Moishezon variety with rational singularity. By the result we proved in the first time, X is a projective variety. □

2.2 Nakai-Moishezon ampleness critera

Proposition 2.2 ([Kol90, Theorem 3.11])**.** Let X be a proper Moishezon space over $\mathbb{C}$ and let H be a line bundle on X. Then H is ample on X if an only if for every irreducible closed subspace $Z \subset X$, the intersection product $H^{\dim(Z)} \cdot Z$ is positive.

2.3 Seshadri criterion line bundle version

Seshadri constant was first introduced by Demailly in the early 90s, when he studied Fujita's conjecture.

Conjecture 2.3. Let X be a smooth projective variety of dimension n, with L being ample. Then

(a) $K_X + (n+1)L$ is global generated,

(b) $K_X + (n+2)L$ is very ample.

Definition 2.4. Given a proper analytic space X and a line bundle L, the *Seshari constant* is defined to be
$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\operatorname{mult}_x C}.$$

He tried to reduce Fujita's conjecture to the bound control of the Seshadri constant.

Theorem 2.5 ([Dem92])**.** Let X be a smooth projective variety of dimension n with L being ample. Then the following hold.
(a) If $\epsilon(L, x) > \frac{n}{n+1}$ then $K_X + (n+1)L$ is global generated,

(b) If $\epsilon(L, x) > \frac{2n}{n+2}$ then $K_X + (n+2)L$ is very ample.

For the readers who want to know more about this, please refer to [Dem92].

Proposition 2.6 ([Kol22])**.** Let X be a proper Moishezon space, and D a divisor on X (the same also true for $\mathbb{Q}$, $\mathbb{R}$ dvisor). Then D is ample if and only if there exists a positive number $\varepsilon > 0$ such that

$$\frac{(D\cdot C)}{\text{mult }_x C} \geq \varepsilon,$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x.

2.4 Seshadri criterion cohomology class version

Lemma 2.7. Let X be a normal compact Moishezon variety. Then the canonical map

$$\Phi : N^1(X) \to N_1(X)^\vee, \quad [D] \mapsto \lambda_D$$

is an isomorphism. Here we define

$$\lambda_D : N_1(X) \to \mathbb{R}, \quad [T] \mapsto T \cdot D.$$

Remark 2.8. For Fujiki varieties with rational singularity the result is also true:

Let X be a normal compact Fujiki variety with rational singularity. Then the canonical map

$$\Phi : N^1(X) \to N_1(X)^\vee, \quad \omega \mapsto \lambda_\omega$$

is an isomorphism. Here we define

$$\lambda_\omega : N_1(X) \to \mathbb{R}, \quad [T] \mapsto T(\omega).$$

Here

$$N^1(X) := H^{1,1}_{\mathrm{BC}}(X),$$

and $N_1(X)$ to be the vector space of real closed currents of bidimension $(1,1)$ modulo the following equivalence relation: $T_1 \equiv T_2$ if and only if

$$T_1(\eta) = T_2(\eta),$$

for all real closed $(1,1)$-forms η with local potentials.

Proposition 2.9 ([Kol22])**.** Let X be a proper Moishezon space over $\mathbb{C}$ with rational singularities. Then X is projective iff there is a cohomology class $\Theta \in H^2(X,\mathbb{Q})$ and an $\epsilon > 0$ such that

$$\Theta \cap [C] \geq \epsilon \cdot \text{mult}_p C$$

for every integral curve $C \subset X$ and every $p \in C$.

Proof. Note that the cup product induce a $\mathbb{Q}$-bilinear form

$$(-) \cap (-) : H^2(X,\mathbb{Q}) \times H_2(X,\mathbb{Q}) \to \mathbb{Q},$$

which will induce a $\mathbb{Q}$-linear functional on $H_2(X,\mathbb{Q})$. If $C \mapsto [C]$ gives an injection $N_1(X,\mathbb{Q}) \hookrightarrow H_2(X,\mathbb{Q})$, then we can view $C \mapsto \Theta \cap [C]$ as a $\mathbb{Q}$-linear map

$$\Theta \cap (-) : N_1(X,\mathbb{Q}) \to \mathbb{Q}.$$

By the previous lemma, $\Theta \cap (-)$ lies in the dual space $N^1(X, \mathbb{Q})$. And line bundles span the dual space of $N_1(X, \mathbb{Q})$. So there is a line bundle L on X and an $m > 0$ such that $\deg(L|_C) = m \cdot \Theta \cap [C]$ for every integral curve $C \subset X$. Thus

$$\deg(L|_C) = m \cdot \Theta \cap [C] \geq m\epsilon \cdot \mathrm{mult}_p C,$$

for every integral curve $C \subset X$ and every $p \in C$. Then L is ample by the line bundle version Seshadri criterion. Therefore X is projective.

Note $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$ if X has 1-rational singularities has been discussed in the first note. □

2.5 Klieman's ampleness criterion for Moishezon spaces

Proposition 2.10 ([VP21]). Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities and that L is a Cartier divisor on Y. Then L is ample if and only if L has positive degree on every irreducible curve on Y and L induces a strictly positive function on $\overline{\mathrm{NE}}(Y)$.

Remark 2.11. It remains open if the result is still true without the $\mathbb{Q}$-factorial KLT assumption.

Proof. The proof require the study of rational curves on Moishezon spaces, we will prove it in the next note. □

3 Approximation of the Chow-Barlet 1-cycle space

In this section, we will introduce the main technical tool: Chow-Barlet cycle space. We will proved that one can approximate the Chow-Barlet 1-cycle space using countable many families of marked curves, which is crucial for the proof of result Theorem 6.1.

Definition 3.1 (Chow functor with m-marked points, [Kol96, Definition I.3.20]). Let X be an analytic space over S. Let

$$\mathrm{Chow}_m(X/S)(Z) = \left\{ \begin{array}{c} \text{Well defined families of nonnegative,} \\ \text{proper, algebraic cycles } \mathcal{C} \text{ of } X \times_S Z/Z, \\ s_1, \ldots, s_m : Z \to X, s_i(z) \in \mathcal{C}_z \text{ for all } z \in Z \end{array} \right\}.$$

We call the data in the bracket the *Chow data with m-marked points*. We say C is a *pointed curve* if it is a 1-cycle that has one marked point. And we denote the Barlet-Chow 1-cycle space with 1-marked point $\mathrm{Chow}^1_1(X/S)$.

Lemma 3.2 (Representative of the Chow functor with marked points). Let $X \to S$ be a proper morphism between complex analytic spaces. The relative Chow functor with m-marked points is representable by a complex analytic space $\mathrm{Chow}_m(X/S)$.

Proof. Since the proof does not appear in the standard references, for the completeness we add a proof here. We claim that Chow functor with marked points is actually represented by a closed subspace of the original Chow-Barlet cycle space (we call this closed subspace incident complex subspace). Let

$$\mathcal{U} \to \mathrm{Chow}(X/S),$$

be the universal family of the Barlet-Chow cycle space (with $\mathcal{U} \subset X \times_S \mathrm{Chow}(X/S)$ as closed complex subspace). We then define the m-fold fiber product to be $X^{(m)} = \underbrace{X \times_S X \times_S ... \times_S X}_{m\text{-times}}$.

Let $P = \mathrm{Chow}(X/S) \times_S X^{(m)}$, the incident complex subspace is defined to be

$$\mathrm{Chow}_m(X/S) = I = \{(s, x_1, ..., x_m) \in P \mid x_i \in \mathcal{U}_s, \text{ for all } i\}.$$

We claim that $I \subset P$ is a closed complex subspace. Indeed, we have the natural projective

$$p_i : P \to \mathrm{Chow}(X/S) \times_S X, \quad (c, x_1, ..., x_m) \mapsto (c, x_i),$$

and easy to check that the incidence variety can be represented as

$$I = \bigcap_{i=1}^{m} p_i^{-1}(\mathcal{U}),$$

since $\mathcal{U}$ is closed complex subspace in $X \times_S \mathrm{Chow}(X/S)$, and therefore as a finite intersection I is a closed complex subspace in P.

We then show that I is the representative of the Chow functor with marked points that is

$$\mathrm{Hom}_S(T, I) \simeq \mathrm{Chow}_m(X/S)(T).$$

To see this, we first show that given a S-morphism $T \to I/S$ it will induce a Chow data with marked points over S. Indeed, since $I \subset \mathrm{Chow}(X/S) \times_S X^{(m)}$, so that the first projection

$$\pi_1 : T \to I \to \mathrm{Chow}(X/S),$$

will induce a family over T via pull back. And the second projection

$$\sigma_i = \pi_{2,i} : T \to I \xrightarrow{q_i} X,$$

will defines the section we want. Conversely, given the Chow data $(\mathcal{Z}, \sigma_1, ..., \sigma_m)$ with marked point, it will induce a morphism. To see this, by the representative of the standard Chow functor, we know that there exists a morphism $\phi_{\mathcal{Z}} : T \to \mathrm{Chow}(X/S)$ such that $\mathcal{Z} \to T$ is the pull back family, with m-sections $\sigma_i : T \to X^{(m)}$. It is easy to check that the induced morphism actually maps into I,

$$\phi_{\mathcal{Z}} \times \sigma_i : T \to I \subset P.$$

□

The following upper semi-continuity result is needed in the proof.

Lemma 3.3 (upper semi-continuity of the multiplicities, [BM19, Proposition 4.3.10])**.** Let $(X_s)_{s \in S}$ be an analytic family of n-cycles of a complex space M. Then the function

$$S \times M \longrightarrow \mathbb{N}, (s, z) \mapsto \mathrm{mult}_z(X_s)$$

is upper semicontinuous in the Zariski topology of $S \times M$.

Proof. The proof of the lemma is a bit complicated and we omit it here. □

Remark 3.4. In particular, let $f : X \to S$ be a proper flat morphism of relative dimension 1, assume that there is a holomorphic section $\sigma : S \to X$. Then the multiplicity

$$\mathrm{mult} : S \to \mathbb{Z}, \quad s \mapsto \mathrm{mult}_{\sigma(s)} X_s$$

is Zariski upper-semicontinuous.

Proof. Since the fibers $\{X_s\}$ clearly forms an anlytic family of cycles in X. Since the section map $\sigma : S \to X$ is holomorphic,

$$S \to S \times X \to \mathbb{N}, \quad s \mapsto (s, \sigma(s)) \mapsto \mathrm{mult}_{\sigma(s)} X_s,$$

is upper semi-continuous. □

Theorem 3.5 (Approximation Chow-Barlet 1-cycle space, [Kol22])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces that is bimeromorphic to a projective morphism. Fix $m \in \mathbb{N}$. Then there are countably many diagrams of complex analytic spaces over S,

$$\begin{array}{ccc} C_i & \hookrightarrow & W_i \times_S X \\ {\scriptstyle w_i}\downarrow\uparrow{\scriptstyle \sigma_i} & & \\ W_i & & \end{array}$$

indexed by $i \in I$, such that

(1) the $w_i : C_i \to W_i$ are proper, of pure relative dimension 1 and flat over a dense, Zariski open subset $W_i^\circ \subset W_i$,

(2) the fiber of w_i over any $p \in W_i^\circ$ has multiplicity m at $\sigma_i(p)$,

(3) the W_i are irreducible, the structure maps $\pi_i : W_i \to S$ are projective, and

(4) the fibers over all the W_i° give all irreducible curves that have multiplicity m at the marked point.

Proof. By assumption, there is a bimeromorphic morphism $r : Y \to X$ such that Y is projective over S.

$$\begin{array}{ccc} Y & \xrightarrow{\quad r \quad} & X \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

By Lemma, the Barlet-Chow cycle space of curves with marked points on Y/S exists (denote it $\mathrm{Chow}_1^1(Y/S)$) and its irreducible components W_i are projective over S. The universal family

$$\mathcal{C} \to \mathrm{Chow}_1^1(Y/S),$$

parameterize all pointed curves on Y in the fiber direction. Let W be any irreducible component of $\mathrm{Chow}_1^1(Y/S)$, We restrict the universal family on that component $C^Y \to W$.

We then map back the family of curves on Y:

$$\begin{array}{ccc} C^Y & \hookrightarrow & W \times_S Y \\ w_Y \big\downarrow \big\uparrow \sigma_Y & & \\ W & & \end{array}$$

to family of curves on X:

$$\begin{array}{ccc} C & \hookrightarrow & W \times_S X \\ w \big\downarrow \big\uparrow \sigma & & \\ W & & \end{array}$$

(Note that the family $w : C \to W$ is no longer flat, as curves on the fibers can be contracted by $Y \to X$).

However, it's still proper flat over some dense Zariski open subset $W^o \subset W$. Since the family is flat over W^o, by Lemma 3.4, the multiplicity of a fiber C_w at the section s is an upper semi-continuous function on W°. For each $m \in \mathbb{N}$, let $W^m \subset W$ be the closure of the set of points $p \in W^\circ$ for which $\mathrm{mult}_{\sigma(p)} C_p = m$. Since the restriction of a projective morphism over closed subvariety is still projective, $W^m \to S$ is a projective morphism.

We finally going back to the original Moishezon morphism $g : X \to S$. Let $X^\circ \subset X$ be the largest open set over which $r : Y \to X$ is an isomorphism. The above procedure gives all irreducible pointed curves that have nonempty intersection with X°. Equivalently, all curves with a marked point that are not contained in $X \backslash X^\circ$. We can now use dimension induction (Note that by the result we proved in the first time the restriction $X \setminus X^0 \to S$ is a Moishezon morphism, so that we can repeat the same argument). And we can get countably many families of pointed curves that approximate the Chow-Barlet 1-cycle space with 1-marked point. □

4 Projectivity of very general fibers

We can now prove the following theorem, which is the key step in deducing the main result.

Theorem 4.1 (Projectivity of very general fibers, [Kol22, Proposition 14])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that
(1) X_0 is projective for some $0 \in S$,
(2) the fibers X_s have rational singularities for $s \in S^*$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Euclidean open neighborhood $0 \in U \subset S$ and countably many nowhere dense, closed, analytic subsets $\{H_j \subset U : j \in J\}$, such that X_s is projective for every $s \in U \setminus \cup_j H_j$.

Proof. First choose $0 \in U \subset S$ such that X_U retracts to X_0. Since X_0 is projective, it carries an ample line bundle L. Let $\Theta \in H^2(X_U, \mathbb{Q})$ be the pull-back of $c_1(L)$ to X_U. Note that Θ is a topological cohomology class that is usually not the Chern class of a holomorpic line bundle. Let (C_s, p_s) be any marked curve on the fiber X_s for $0 \neq s \in U$.

Using Theorem 3.5, we can find countable many families of pointed curves, with projective morphisms $\pi_i : W_i \to U$.

$$\begin{array}{ccc} C_i & \hookrightarrow & W_i \times_S X \\ {\scriptstyle w_i}\big\downarrow\big\uparrow{\scriptstyle \sigma_i} & & \\ W_i & & \end{array}$$

Let $J \subset I$ be the index such that $H_i := \pi_i(W_i) \subset U$ for $i \in J$ is nowhere dense in U. Therefore, $\pi_i : W_i \to U$ for $i \in I \setminus J$ will dominant U. Since π_i is projective, in particular it implies $0 \in \pi_i(W_i)$ for $i \in I \setminus J$.

Let $s \in U \setminus \cup_{j \in J} H_j$, then by definition of J, there is an $i \in I \backslash J$, such that the following conditions hold.

(a) (C_s, p_s) is one of the fibers of w_i over W_i°,

(b) $\text{mult}_{\sigma_i(p)} C_p = m$ for all $p \in W_i^\circ$, and

(c) $\pi_i : W_i \to U$ is projective and its image contains $0, s \in S$ (say $\pi_i(0) = 0, \pi_i(w) = s$)

Since W_i is irreducible, there exist a holomorphic curve $\tau : \Delta \to W_i$ connecting the point $0, w$ (with $\tau(0) = 0, \tau(1) = w$ and the radius of $r(\Delta) \ > 1$). We then pull the family back to the disc

$$w : \mathcal{C} \to \Delta,$$

with section $\sigma : \Delta \to \mathcal{C}$. Note that

$$\text{mult}_{\sigma(t)} \mathcal{C}_t = \text{mult}_{\sigma(1)} \mathcal{C}_1 = \text{mult}_{\sigma_i(s)} C_{is} \text{ for all } t \in \Delta^*,$$

since $\tau(\Delta^*) \subset W_i^o$. On the other hand, by the Lemma 3.4, we have

$$\text{mult}_{\sigma(0)} \mathcal{C}_0 \geq \text{mult}_{\sigma(t)} \mathcal{C}_t = \text{mult}_{\sigma_i(s)} (C_i)_s, \quad \text{for } t \in \Delta^*.$$

(Here the pull back family $\mathcal{C} \to \Delta$ is flat, since the base is a disc and a surjective holomorphic map from reduced irreducible space to a disc is automatically flat).

Since $\mathcal{C}_0$ is a 1-cycle on the projective X_0, and $\Theta_0 = \Theta|_{X_0}$ is the Chern class of an ample line bundle on X_0. Thus

$$\Theta \cap [\mathcal{C}_0] \geq \epsilon \cdot \text{mult}_{\sigma(0)} \mathcal{C}_0.$$

by the easy direction of Theorem 2.9, where ϵ depends only on X_0 and Θ_0.

Since $\mathcal{C}_0$ and $\mathcal{C}_1$ lie in the same irreducible component of Chow-Barlet cycle space, they are algebraic equivalent. Thus the cup product with Θ remain the same. Putting these together gives that

$$\boxed{\Theta_s \cap [C_s] = \Theta \cap [\mathcal{C}_1] = \Theta \cap [\mathcal{C}_0] \geq \epsilon \cdot \text{mult}_{p_0} \mathcal{C}_0 \geq \epsilon \cdot \text{mult}_{p_s} C_s}$$

Thus X_s is projective by another direction of Theorem 2.9. $\square$

5 From locally projective to global projective

Lemma 5.1 (Trivialization of the monodromy after finite base change)**.** Let X be a connected complex analytic variety. Let $\mathscr{L}$ be a local system with finite monodromy defined on X. Then there exisfts a finite covering $\pi : X' \to X$ such that the pull back local system $\pi^*\mathscr{L}$ becomes trivial.

Proof. Let

$$\rho : \pi_1(X, x_0) \to \mathrm{GL}_n(L),$$

be the monodromy representation, with L be the fiber of the local system at the reference point $x_0 \in X$. Since the monodromy of $\mathscr{L}$ is finite, so that

$$\ker \rho \subset \pi_1(X, x_0),$$

is a finite index normal subgroup. Thus by the Galois correspondence, we can find a finite cover

$$\pi : X' \to X,$$

such that the fundamental group $\pi_*(\pi_1(X', x'_0)) = \ker \rho \subset \pi_1(X, x_0)$ with $\pi(x'_0) = x_0$. On the other hand, we have the following base change diagram for the monodromy representation.

$$\begin{array}{ccc} \pi_1(X', x'_0) & \xrightarrow{\pi_*} & \pi_1(X, x_0) \\ {\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathrm{GL}(L) & \xrightarrow[=]{} & \mathrm{GL}(L) \end{array}$$

so that the monodromy of $\rho' : \pi_1(X', x'_0) \to \mathrm{GL}(L)$ is clearly trivial. □

Lemma 5.2. Let $g : X \to Y/S$ be a proper contraction morphism defined over S. The induced pull back map on the Néron-Sever group and N^1 space

$$g^* : \mathrm{NS}(Y/S) \to \mathrm{NS}(X/S), \quad g^* : N^1(Y/S) \to N^1(X/S),$$

are injective.

Proposition 5.3. Assume that $g : X \to S$ be a proper Moishezon morphism of normal irreducible analytic spaces. Assume that there exists a dense Zariski open subset $S^o \subset S$ such that X is locally projective over S^o then it's actually global projective.

Proof. By passing to a Zariski open subset, we may assume that $R^2 g_* \mathcal{O}_X$ is locally free, and $R^2 g_* \mathbb{Z}_X$ is locally constant. Thus by Proposition **??** and Lemma 5.2, after finite base change the Neron-Sever local system becomes trivial local system, i.e. the locally defined ample line bundle

$$L_i \in \mathcal{NS}(X/S)(U_i) = \mathcal{NS}(X/S)(S),$$

defines a global line bundle. □

6 Kollár's projective stratification theorem

Now we can prove the main theorem of this note.

Theorem 6.1 (Projective Stratification, [Kol22, Theorem 2])**.** Let $g : X \to S$ be a proper Moishezon morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,

(2) the fibers X_s have rational singularities for $s \in S^*$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \text{ is projective.}$$

Proof. By Theorem 4.1, we know that $\mathrm{PR}_S(X)$ contains the complement of a countable union of Zariski closed, nowhere dense subsets. By the Baire category theorem, $\mathrm{PR}_S(X)$ is not contained in a countable union of closed, nowhere dense subsets. And by the alternating property of projective locus that we proved in the previous note, we are in the case that $g : X \to S^o$ is locally projective over a dense, Zariski open subset $S^\circ \subset S$.

Since the morphism is Moishezon, therefore by [Kol22, Complement 18], the morphism $g : X \to S$ is global projective over S^o. And we repeat the process on $S \setminus S^o$ gives the stratification of $g : X \to S$ into projective morphisms $g|_{X_i} : X_i = g^{-1}(S_i) \to S_i$. □

7 Claudon-Höring's projectivity criterion for Kähler morphisms

In this section, we introduce the following projectivity criterion for Kähler morphism.

Theorem 7.1 ([CH24, Theorem 3.1])**.** Let $f : X \to Y$ be a fibration between normal compact Kähler spaces. Assume that X has strongly $\mathbb{Q}$-factorial KLT singularities. Assume one of the following:
(1) The normal space Y has klt singularities and the natural map

$$f^* : H^0\left(Y, \Omega_Y^{[2]}\right) \longrightarrow H^0\left(X, \Omega_X^{[2]}\right)$$

is an isomorphism.
(2) The morphism f is Moishezon.

Then f is a projective morphism.

Proof. We will discuss this in the next note. □

Final words, Projectivity of moduli has been systematic studied by Kollár in the 1990's. For readers who want to know more about this direction, please refer to [Kol90].

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