PhD Dissertation Proposal Examination

Moishezon space and Moishezon morphism

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Moishezon variety

Definition (Moishezon variety)

A proper, irreducible, reduced analytic space X is Moishezon if it is bimeromorphic to a projective variety X^{p}.

Remark

There are several equivalent definitions:

- *A proper irreducible, reduced analytic space X is Moishezon if there exist a birational modification $X^p \to X$ with X^p being projective.*
- *A Moishezon variety is a complex variety such that*

$$a(X) := tr\deg_{\mathbb{C}} M(X) = \dim(X)$$

  that is it has $\dim X$ number of algebraic dependent meromorphic function.
- *A Moishezon manifold is a complex manifold that equipped with a big line bundle.*

Moishezon morphism

Definition (Moishezon morphism)

Assume now that S is reduced. A proper morphism of analytic spaces $g : X \to S$ is Moishezon if $g : X \to S$ is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Remark

The definition above is equivalent to the followings

- *A proper morphism of analytic spaces $g : X \to S$ is Moishezon if There is a projective morphism of algebraic varieties $G : \mathrm{X} \to \mathrm{S}$ and a meromorphic $\phi_S : S \dashrightarrow \mathrm{S}$ such that X is bimeromorphic to $\mathrm{X} \times_{\mathrm{S}} S$.*
- *Assume now that S is reduced. A proper morphism of analytic spaces $g : X \to S$ is Moishezon if there is a rank 1, reflexive sheaf L on X such that the natural map $X \dashrightarrow \mathrm{Proj}_S(g_* L)$ is bimeromorphic onto the closure of its image.*

Properties of Moishezon morphism

For more properties, see [Kol22a] and [Fuj83].

Theorem (Morphism on exceptional locus is Moishezon)

Let $g : X \to S$ be a proper, generically finite, dominant morphism of normal, complex, analytic spaces. Then $\mathrm{Ex}(g) \to S$ is Moishezon.

Theorem (Base change property for Moishezon morphism)

The fibers of a proper, Moishezon morphism are Moishezon.

Moishezon locus

Moishezon locus satisfies alternating property.

Theorem (Rao-Tsai 22')

Theorem 21. Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces. Then the Moishezon locus $\mathrm{MO}_S(X) \subset S$ is
(1) either contained in a countable union $\cup_i Z_i$, where $Z_i \subsetneq S$ are Zariski closed,
(2) or $\mathrm{MO}_S(X)$ contains a dense, open subset of S.
Furthermore, if $R^2 g_* \mathcal{O}_X$ is torsion free then (2) can be replaced by
(3) $\mathrm{MO}_S(X) = S$ and g is locally Moishezon.

Fiberwise bimeromorphism

When the central fiber is not uniruled, we can expect the Moishezon morphism to be fiberwise bimeromorphic to a projective morphism.

Theorem (Kollár 22a', Theorem 28)

Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 has log terminal singularities and X_0 is not uniruled.
Then g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ (possibly over a smaller disc) such that

- *X_0^{p} has log terminal singularities,*
- *X_s^{p} is not uniruled and has terminal singularities for $s \neq 0$, and*
- *$K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.*

Algebraic approximation

Given a Moishezon morphism over $\mathbb{D}$, we can replace it by a Moishezon morphism over smooth algebraic curve (in the infinitesimal neighborhood).

Theorem (Algebraic approximation for Moishezon morphism, Kollár 22a', Proposition 37)

Let $f : X \to \mathbb{D}$ be a Moishezon morphism, with the projective modification $h : Y \to X$ over $\mathbb{D}$. Then we can change the diagram to a smooth algebraic curve (C, c). such that the morphism are isomorphic in the m-th order infinitesimal neighborhood of the special fiber

$$(Y \to X)_m \cong (Y_C \to X_C)_m.$$

Inversion of adjunction

The inversion of adjunction holds for Moishezon morphism.

Theorem (Kollár 22a', Theorem 40)

Let $g : X \to \mathbb{D}$ *be a flat, proper, Moishezon morphism and* Δ *an effective* $\mathbb{Q}$*-divisor on* X*. Assume that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier. Then*

$$\operatorname{discrep}(X, X_0 + \Delta_0) = \textit{totaldiscrep}\,(X_0, \Delta_0)$$

where on the left we use only those exceptional divisors whose centers on X *have nonempty intersection with* X_0*.*

Openess of projectivity

Openess of projectivity

Theorem (Kollár 22b', Theorem 1)

Theorem 1. Let $g : X \to \mathbb{D}$ be a proper, flat morphism of complex analytic spaces. Assume that

- *X_0 is projective,*
- *the fibers X_s have rational singularities for $s \neq 0$, and*
- *g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$*

Then g is projective over a smaller punctured disc $\mathbb{D}_\epsilon^\circ \subset \mathbb{D}$.

Remark

- *Without Moishezon assumption, the theorem above is not true.*
- *The proof use Seshadri projectivity criterion and Chow-Barlet cycle space arguement.*

Deformation invariance of plurigenera

Deformation invariance of plurigenera

Theorem (Kollár 21', Theorem 1)

Theorem 1. Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Fix a point $0 \in S$ and assume that the fiber X_0 is projective, of general type, and with canonical singularities. Then there is an open neighborhood $0 \in U \subset S$ such that

- *the plurigenera of X_s are independent of $s \in U$ for every r, and*
- *the fibers X_s are projective for every $s \in U$.*

Remark

- *The proof use the extension of MMP combined with BCHM.*
- *When X_0 does not have canonical singularity, or not Kaehler, the result is not necessarily true.*

Projectivity critera

Conjecture (Peternell's conjecture)

Any Moishezon manifold without rational curves is projective.

Theorem (Paz 21', Corollary 1.3)

Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities. Then Y is non-projective if and only if it contains a rational curve C such that $-[C] \in \overline{\mathrm{NE}}(Y)$.

Remark

- *The proof of Paz use MMP arguement over the Moishezon base.*

Thanks for your attention!