Kähler Minimal Model Program with Applications to Deformation Problems

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Problem-1: Deformation Invariance of Plurigenera

Theorem (Siu 98', Siu 02')

Let $\pi : X \to \Delta$ be a smooth projective family. Then for any positive integer m

$$t \longmapsto h^0\left(X_t, mK_{X_t}\right) := \dim H^0(X_t, mK_{X_t})$$

is independent of t for $t \in \Delta$.

There are several remarkable works after Siu, including: Kawamata 99', Păun 07', Takayama 07', Hacon–McKernan–Xu 13', Li-Wang 26', Rao–Tsai 22', etc.

Conjecture (Siu 02')

Let $\pi : X \to \Delta$ be a smooth Kähler family. Then for any positive integer m

$$t \longmapsto h^0\left(X_t, mK_{X_t}\right) := \dim H^0(X_t, mK_{X_t})$$

is independent of t for $t \in \Delta$.

Problem-2: Deformation Invariance of Volume

As an transcendental analogue of plurigenera, volume exhibits the same deformation invariance properties.

Theorem (Siu 02')

Let $\pi : X \to \Delta$ be a smooth projective family. Then the volume function $t \longmapsto vol(K_{X_t})$ is independent of $t \in \Delta$.

When we consider transcendental classes, the plurigenera and Kodaira dimension are not defined, but we can still consider the volume.

Conjecture (Hacon-L.-Rao 26')

Let $f : X \to \Delta$ be a smooth family from a Kähler manifold. Assume that $(X, B + \boldsymbol{\beta})$ is a generalized klt pair such that (X, B) is log smooth over Δ and $\boldsymbol{\beta} = \overline{\beta}$ where β is nef over Δ. Then the volume function

$$t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \boldsymbol{\beta}_{X_t})$$

is constant over Δ.

Problem-3: Deformation of Uniruledness

Theorem (Fujiki–Levine 82')

Let $f : X \to S$ be a smooth Kähler family over a smooth, connected, and relatively compact curve S. If X_0 is uniruled, then X_t are uniruled for $t \in S$.

Fujiki–Levine's method tries to spread a **free rational curve** from the central fiber to the nearby fibers, consequently showing that the nearby fibers are uniruled. Note that free rational curves on singular varieties are tricky, their method not directly applicable to the singular case.

Conjecture (Hacon-L.-Rao 26')

Let $f : X \to S$ be a Kähler family over a smooth, connected, and relatively compact curve S. Assume that X_t have canonical singularities for all $t \in S$. Then X_0 is uniruled if and only if X_t are uniruled for all $t \in S \setminus \{0\}$.

Problem-4: Deformation of Pseudo-effectiveness

As a consequence of Fujiki–Levine 82' and BDPP 13' (a smooth projective manifold is uniruled if and only if K_X is not pseudo-effective), we have the pseudo-effectiveness in family:

Theorem (Tsuji 02')

Let $\pi : X \to \Delta$ be a smooth projective family. If K_{X_0} is pseudo-effective, then K_{X_t} is pseudo-effective for $t \in \Delta$.

Conjecture (Hacon-L.-Rao 26')

Let $f : X \to S$ be a Kähler family onto a smooth, connected, and relatively compact curve S. Assume that X_t have canonical singularities for all $t \in S$. Then K_{X_0} is pseudo-effective if and only if K_X is pseudo-effective over S.

Our Main Results (Hacon–L.–Rao, 2026)

Let $f: X \to S$ be a Kähler family over a smooth curve S, X_0 projective.

✓	**Uniruled** (Prob. 4)	*canonical singularities*	X_0 uniruled $\Leftrightarrow$ X_t uniruled
✓	**Pseudo-eff.** (Prob. 3)	*canonical singularities*	K_{X_0} psef $\Leftrightarrow$ K_X psef over$S$
✓	**Volume** (Prob. 2)	(X, B) *log smooth*, $K_{X_0} + B_0 + \beta_0$ *big*	$\mathrm{vol}(K_{X_t} + B_t + \beta_t)$ const.
✓	**Plurigenera** (Prob. 1)	*smooth*, $\dim = 3$	$P_m(X_t)$ const., $\forall\, m \geq 1$

By the Kähler threefold MMP, we are able to prove all the results above without assuming X_0 is projective.
Key engine: Kollár's MMP extension technique.

Preliminaries: Positivities

Definition (Pseudo-effectiveness, Nef Classes, Big Classes)

Let X be a normal compact complex analytic variety, and let $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ be a $(1,1)$-class.

1. *The class α is called* ***metrically nef*** *if, for some positive $(1,1)$-form ω on X, for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{A}^0(X)$ such that*

$$\alpha + i\partial\bar{\partial} f_\varepsilon \geq -\varepsilon\,\omega.$$

2. *The class α is* ***pseudo-effective*** *if it contains a semi-positive current $T \geq 0$.*
3. *The class α is called* ***big*** *if there exists a positive $(1,1)$-form ω on X such that $\alpha - \{\omega\}$ is pseudo-effective.*

A class α is called **algebraically nef** if $\alpha \cdot C \geq 0$ for all irreducible curves. Metrically nef implies algebraically nef. In the Kähler MMP, we use metrically nef instead of algebraically nef.

Relative Pseudo-effective, nef, big

Definition (Relative Pseudo-effective, Nef, Big)

Let $f : X \to S$ be a proper surjective morphism from a normal complex analytic variety X onto a relatively compact base S, and let $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ be a $(1,1)$-class. We say that:

1. *The class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is f-**nef** or relatively nef over S if for any $s \in S$, the restriction $\alpha_s := \alpha|_{X_s}$ is a nef class on the fiber X_s.*
2. *The class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is f-**pseudo-effective over** S or relatively pseudoeffective over S if there is a complement V of a countable union of Zariski closed proper subsets of S such that $\alpha_s := \alpha|_{X_s}$ is pseudo-effective for any $s \in V$.*
3. *The class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ is f-**big over** S or relatively big over S if there is a non-empty Zariski open subset U of S such that α_s is big for any $s \in U$.*

Preliminaries: Generalized Pairs

Generalized pairs were first introduced by Birkar–De-Qi Zhang 16' in the study of the Effective Iitaka Fibration Conjecture. Some special cases were already investigated in Bikar-Hu 14'.

Definition (Generalized Kähler Pairs, Das-Hacon-Yáñes 25')

Let $f : X \to S$ be a proper surjective morphism from a normal complex variety X to a relatively compact complex variety S, $\nu : X' \to X$ a resolution, and B' an $\mathbb{R}$-divisor on X' with simple normal crossing support such that $B := \nu_* B' \geq 0$, and β a closed b-$(1,1)$ current. We say that $(X, B + \beta)$ is a generalized pair if

(1) β is a positive closed b-$(1,1)$ current that descends to X',

(2) $[\beta_{X'}] \in H^{1,1}_{\mathrm{BC}}(X', \mathbb{R})$ is nef over S, and

(3) $[K_{X'} + B' + \beta_{X'}] = \nu^* \gamma$ for some $\gamma \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$.

Recently, Jia Jia and Meng Sheng 25' construct an example, showing that nef cone of some **non-Kähler** variety is 0, generalized pairs may not well defined in the non-Kähler setting.

Preliminaries: Volumes of cohomology classes

Definition (Volume of cohomology classes)

For $\alpha \in \mathrm{H}^{1,1}_{\mathrm{BC}}(\mathrm{X}, \mathbb{R})$ on a normal compact Kähler variety X, if α is a pseudo-effective cohomology class, we define the volume of α by pulling back via a resolution $\pi : \mathrm{X}' \to \mathrm{X}$

$$\mathrm{vol}(\alpha) := \sup_{\mathrm{T} \in \pi^*(\alpha)} \int_{\mathrm{X}'} \mathrm{T}^{\mathrm{n}}_{\mathrm{ac}} = \sup_{\mathrm{T} \in \pi^*(\alpha)} \int_{\mathrm{X}' \setminus \mathrm{Sing}(\mathrm{T})} \mathrm{T}^{\mathrm{n}},$$

Here T_{ac} denotes the absolutely continuous part of the current T, and $Sing(T)$ denotes the singular locus of T.

Remark. Given a pseudo-effective line bundle L on a compact Kähler manifold X, Boucksom 02' showed that the usual definition of the volume of a line bundle,

$$\mathrm{vol}(L) := \limsup_{k \to +\infty} \frac{n!}{k^n} h^0(X, kL),$$

coincides with definition above.

Preliminaries: Projective MMP

For projective varieties of log general type the minimal model theory is completely known.

Theorem (BCHM 10', Birkar–De-Qi Zhang 16', Das–Hacon 24', Hacon-L.-Rao 26')

Let $(X, B+\beta)$ be a compact generalized klt pair, and $f: X \to S$ a projective morphism over a relatively compact base such that $B+\beta_X$ is big over S. Then the following hold:
(1) if $K_X + B + \beta_X$ is pseudo-effective over S, then $(X, B+\beta)$ has a good minimal model over S, and
(2) if $K_X + B + \beta_X$ is not pseudo-effective over S, then $(X, B+\beta)$ has a Mori fiber space over S.

Kähler MMP — Major Challenges

It is conjectured that the Minimal Model Program (MMP) extends to the Kähler setting. However, there are several fundamental challenges:

- Mori's Bend-and-Break technique,
- Transcendental base point freeness,
- Contraction theorem,
- Existence and termination of the MMP.

Fortunately, Recent developments in Kähler MMP have allowed us to address these problems in certain special cases.

Cone and contraction Theorem

Hacon–Păun 25' established the Kähler cone theorem in arbitrary dimension.

Theorem (Cone Theorem, Hacon–Păun 25')

Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B+\beta)$ is generalized klt. Then there are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\beta_X\geq 0} + \sum_{i\in I} \mathbb{R}^+ [\Gamma_i]$$

where $0 < -(K_X + B + \beta_X) \cdot \Gamma_i \leq 2n$.

Cone and contraction Theorem

In the classical MMP, the contraction theorem is a consequence of the **Kawamata base point free theorem**. In the Kähler case, however, the transcendental base point free theorem remains an open problem, and hence so does the contraction theorem:

Conjecture (Contraction Theorem)

Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B+\beta)$ is generalized klt. Let F be a $(K_X+B+\beta)$-negative extremal face of $\overline{NA}(X)$. Then there exists a contraction morphism $h: X \to Y$ associated to F, such that a curve C is contracted if and only if $C \in F$, and $-(K_X+B+\beta)$ is h-Kähler.

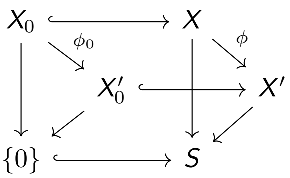
Due to the lack of a contraction theorem, we cannot run the full Kähler MMP.

Extension of Contraction

Fortunately, Kollár's extension of contraction technique allows us to implement contractions in the Kähler setting.

Theorem (Kollár–Mori 92')

Let $\phi_0 : X_0 \to X_0'$ be a proper surjective morphism of normal compact complex analytic spaces. Assume that $(\phi_0)_*\mathcal{O}_{X_0} = \mathcal{O}_{X_0'}$ and $R^1(\phi_0)_*\mathcal{O}_{X_0} = 0$. If $f : X \to S$ is a small deformation of X_0, then ϕ_0 can be extended to a contraction morphism $\phi : X \to X'$ over S.



Here $R^1(\phi_0)_*\mathcal{O}_{X_0} = 0$ indicates the vanishing of the obstruction againest extending the morphism.

Extension of Contraction

To check the vanishing of the obstruction againest extending the morphism $R^1(\phi_0)_*\mathcal{O}_{X_0} = 0$, we require the following relative version transcendental Kawamata–Viehweg vanishing theorem:

Theorem (Hacon-L.-Rao 26')

Let $g : X \to S$ be a proper surjective morphism from a generalized Kähler pair $(X, B + \beta)$ with generalized klt singularities onto a relatively compact complex analytic variety S. If D is a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on X such that $D - (K_X + B + \boldsymbol{\beta}_X)$ is relatively nef and big over S, then $R^i g_*\mathcal{O}_X(D) = 0$ for all $i > 0$.

Extension of MMP

Theorem (Kollár21', Hacon-L.-Rao 26')

Let $g : X \to S$ be a flat, proper contraction morphism from a generalized Kähler pair $(X, B + \beta)$. Suppose that $(X_0, B_0 + \beta_0)$ is normal $\mathbb{Q}$-factorial, projective, with canonical singularities, and

$$N(K_{X_0} + B_0 + \beta_0) \wedge B_0 = 0.$$

Then every sequence of $(K_{X_0} + B_0 + \beta_0)$-transcendental MMP steps

$$X_0 \dashrightarrow X_0^{(1)} \dashrightarrow X_0^{(2)} \dashrightarrow \dots$$

extends to a sequence of $(K_X + B + \beta_X)$-negative proper meromorphic maps

$$X/U \dashrightarrow X^{(1)}/U \dashrightarrow X^{(2)}/U \dashrightarrow \dots,$$

over some open neighborhood $U \subset S$ of 0.

Stability of Big and Nef

Our stability of big and nef result make it possible to propagates the termination from the central fiber to nearby fibers.

Theorem (Hacon-L.-Rao 26')

Let $f : X \to S$ be a proper surjective morphism from a normal $\mathbb{Q}$-factorial generalized Kähler pair $(X, B + \boldsymbol{\beta})$ onto a smooth, connected, relatively compact curve S, and ω a Kähler form on X. Fix a point $0 \in S$ and assume that the support of the boundary divisor B does not contain the fiber X_0.
Assume that the restriction to the central fiber $(X_0, B_0 + \boldsymbol{\beta}_0)$ is a projective, gklt pair such that $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is nef and big. Then $K_X + B + \boldsymbol{\beta}_X$ is nef and big over U for some open neighborhood $U \subset S$ of 0.

The proof of this result relies on the transcendental base point free theorem for projective gklt pairs([Das-Hacon 26']).

Pseudo-effectiveness in Families

Theorem (Hacon-L.-Rao 26')

Let $f : X \to S$ be a family from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Assume that X_0 is projective, and X_t have canonical singularities for all $t \in S$. Then the canonical divisor K_{X_0} is pseudo-effective if and only if K_X is pseudo-effective over S.

Deformation closedness of pseudo-effectiveness (K_{X_t} pseff implies K_{X_0} pseff):

We argue by contradiction. Assume that $K_{X_0} + B_0 + \beta_0$ is not pseudo-effective. The termination result implies that the $(K_{X_0} + B_0 + \beta_0)$-MMP with scaling terminates with a Mori fiber space $X_0' \to Z_0$:

$$X_0 \dashrightarrow X_0^{(1)} \dashrightarrow \cdots \dashrightarrow X_0' \to Z_0.$$

Proof Idea: Pseudo-effectiveness in Families

By our extension of MMP technique, there exists a neighborhood $U \subset S$ of 0 such that the steps of the $(K_{X_0} + B_0 + \beta_0)$-MMP extend to a sequence of $(K_X + B + \beta)$-negative proper meromorphic maps over U:

$$(X, B+\beta)/U \dashrightarrow (X^{(1)}, B^{(1)}+\beta^{(1)})/U \dashrightarrow \cdots \dashrightarrow (X', B'+\beta')/U \dashrightarrow \cdots .$$

Moreover, the Mori fiber space $X_0' \to Z_0$ extends to a relative Mori fiber space $X' \to Z$ over U.
This gives a contradiction, since $K_{X_t'} + B_t' + \beta_t'$ is pseudo-effective for fibers over nearby t. $\square$

Proof Idea: Pseudo-effectiveness in Families

Deformation openness of pseudo-effectiveness (K_{X_0} pseff then K_X psef over some U): Since termination in the pseudo-effective case is unavailable, we argue by contradiction. Suppose that K_X is not f-pseudo-effective over any small open neighborhood $U \subset S$ of 0.

(1) For a Kähler class ω on X, define the pseudo-effective threshold

$$\tau' := \inf\{t > 0 \mid K_X + t\omega \text{ is } f\text{-pseudo-effective over some neighborhood } U \subset S\}.$$

Since K_X is not f-pseudo-effective over any $U \subset S$, we have $\tau' > 0$ (this requires Birkar's pseudo-effective threshold result from the BAB paper).

(2) Let $0 < \tau < \tau'$, and consider the pair $(X_0, \tau\boldsymbol{\omega}_{X_0})$ where $\boldsymbol{\omega}_X = \overline{\omega}$. By assumption K_{X_0} is pseudo-effective, so $K_{X_0} + \tau\omega_{X_0}$ is big. Since X_0 is projective, we can run the transcendental MMP on the central fiber. This MMP terminates at $(X_0^{(n)}, \tau\boldsymbol{\omega}_{X_0^{(n)}})$ with nef $K_{X_0^{(n)}} + \tau\omega_{X_0^{(n)}}$:

$$(X_0, \tau\boldsymbol{\omega}_{X_0}) \dashrightarrow (X_0^{(1)}, \tau\boldsymbol{\omega}_{X_0^{(1)}}) \dashrightarrow \cdots \dashrightarrow (X_0^{(n)}, \tau\boldsymbol{\omega}_{X_0^{(n)}}).$$

Proof Idea: Pseudo-effectiveness in Families (Cont.)

(3) By our extension of MMP technique, there exists a neighborhood $0 \in U \subset S$ such that the $(K_{X_0} + \tau\omega_{X_0})$-MMP extends to a proper $(K_X + \tau\omega_X)$-negative map over U:

$$(X, \tau\boldsymbol{\omega}_X)/U \dashrightarrow (X^{(1)}, \tau\boldsymbol{\omega}_{X^{(1)}})/U \dashrightarrow \cdots \dashrightarrow (X^{(n)}, \tau\boldsymbol{\omega}_{X^{(n)}})/U.$$

(4) Since $K_{X_0^{(n)}} + \tau\omega_{X_0^{(n)}}$ is big and nef, by our big and nef adjoint class in family, after further shrinking U, $K_{X^{(n)}} + \tau\omega_{X^{(n)}}$ is nef (and hence also relatively pseudo-effective) over U. Hence $K_X + \tau\omega_X$ is relatively pseudo-effective over U, giving $\tau' \leq \tau < \tau'$, a contradiction. $\square$

Proof Idea: Pseudo-effectiveness in Families (Cont.)

From local stability to global stability (K_X pseff over U to K_X pseff over S:

(1) By contradiction, suppose that K_X is not relatively pseudo-effective over S, then there is an uncountable set $T \subset S$ such that K_{X_t} is not pseudo-effective for any $t \in T$. Hence there is a family $\mathcal{C}_t$ of K_{X_t}-negative rational curves dominating X_t for each $t \in T$.
Since the relative Douady space has only countably many components, there is a component $\mathcal{D} \subset \mathcal{D}(X/S)$ and an uncountable subset $T' \subset T$ of S, such that

$$\mathcal{C}_t \subset \mathcal{D} \text{ for any } t \in T'.$$

Since S is a relatively compact curve, the identity principle implies $\overline{T'} = S$ (where $\overline{T'}$ is the Zariski closure of T').

(2) Let $\mathcal{C} = \bigcup_{s\in T'} \mathcal{C}_s \subset \mathcal{D}$, and $\overline{\mathcal{C}}$ the Zariski closure of $\mathcal{C}$ in $\mathcal{D}$. We then have the commutative diagram, with $\Phi\colon \mathcal{X} \to X$ surjective.

$$\begin{array}{ccc} \mathrm{Univ}_{\overline{\mathcal{C}}} & \xrightarrow{\Phi} & X \\ \downarrow & & \downarrow f \\ \overline{\mathcal{C}} & \xrightarrow[\pi]{} & S. \end{array}$$

(3) Since K_X is relatively pseudo-effective over U, there exists some general $s \in U$, such that K_{X_s} is pseudo-effective. Then $K_{X_s} + \varepsilon\omega_s$ is big for any $\varepsilon > 0$. Let $[C] \in \overline{\mathcal{C}}$ be a general curve that lies in X_s, whose existence follows from the surjectivity of $\Phi : \mathcal{X} \to X$. Then we have

$$0 < (K_{X_s} + \varepsilon\omega_s) \cdot C.$$

On the other hand, for $t \in T'$

$$K_{X_t} \cdot C' < 0, \quad \text{for } [C'] \in \mathcal{C}_t.$$

Therefore, for $0 < \varepsilon \ll 1$, we deduce that

$$0 < (K_{X_s} + \varepsilon\omega_s) \cdot C = (K_X + \varepsilon\omega) \cdot C = (K_X + \varepsilon\omega) \cdot C' = (K_{X_t} + \varepsilon\omega_t) \cdot C' < 0,$$

a contradiction. □

Uniruledness in Families

Theorem (Hacon–L.–Rao 26')

Let $f : X \to S$ be a family from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Assume that X_0 is projective, and X_t have canonical singularities for all $t \in S$. Then X_0 is uniruled if and only if X_t are uniruled for all $t \in S \setminus \{0\}$.

The proof of uniruledness is very similar to the previous one.

Deformation Invariance of Volume for Adjoint Classes

Theorem (Hacon-L.-Rao 26')

Let $f : X \to S$ be a smooth, proper morphism from a Kähler manifold X onto a smooth, connected, and relatively compact curve S. Assume that $(X, B+\boldsymbol{\beta})$ is a generalized klt pair such that (X, B) is log smooth over S and $\boldsymbol{\beta} = \overline{\beta}$ where β is nef over S, and X_0 is projective. If $K_{X_0} + B_0 + \boldsymbol{\beta}_0$ is big, then the volume function

$$t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \boldsymbol{\beta}_t)$$

is constant on an open neighborhood $U \subset S$ of 0.

Proof Idea: Deformation Invariance of Volume

(1) We run the generalized $(K_{X_0} + B_0 + \boldsymbol{\beta}_{X_0})$-MMP with scaling of ω_{X_0} on the central fiber X_0, and it terminates at some minimal model:

$$X_0 \dashrightarrow X_0^{(1)} \dashrightarrow X_0^{(2)} \dashrightarrow \cdots \dashrightarrow X_0^{(n)} \dashrightarrow X_0'.$$

(2) By our extension of MMP technique (with a bit of efforts, we can assume that $N(K_{X_0} + B_0 + \boldsymbol{\beta}_0) \wedge B_0 = 0$ and does effect the result), after shrinking the base to some open neighborhood $U \subset S$ of 0, we extend the steps of the MMP on X_0 to a sequence of $(K_X + B + \boldsymbol{\beta}_X)$-negative proper bimeromorphic maps over U:

$$X/U \dashrightarrow X^{(1)}/U \dashrightarrow \cdots \dashrightarrow X'/U.$$

(3) Since $K_{X_0'} + B_0' + \boldsymbol{\beta}_{X_0'} + \delta\omega_{X_0'}$ is big and nef, by the big and nef adjoint class in family result, after shrinking U, $K_{X'} + B' + \boldsymbol{\beta}_{X'}$ is relatively nef over U. Hence

$$\mathrm{vol}(K_{X_t'} + B_t' + \boldsymbol{\beta}_{X_t'}) = (K_{X_t'} + B_t' + \boldsymbol{\beta}_{X_t'})^m,$$

where $m = \dim_{\mathbb{C}}(X/S)$.

Proof Idea: Deformation Invariance of Volume (cont.)

(4) By step (3), it suffices to show that

$$t \longmapsto (K_{X'_t} + B'_t + \beta_{X'_t})^m$$

is locally constant in the family. For line bundles, the Hilbert polynomial is constant in a flat family. For cohomology classes in our setting, we apply the singular Stokes' formula:

Theorem (Singular Stoke's formula, Stolzenberg 68')

Let M be a smooth, relatively compact, n-dimensional real manifold with finite n -volume, and let $\overline{\mathrm{M}}$ *be its compactification. Assume that* $\overline{\mathrm{M}}\backslash\mathrm{M}$ *can be decomposed as* $\partial \cup \mathrm{S}$*, where*
(1) (M, ∂) *is a manifold with boundary,*
(2) ∂ *has finite* $(\mathrm{n}-1)$*-volume,*
(3) S is compact with $\mathrm{codim}_{\mathbb{R}}(\mathrm{S}, \mathrm{M}) \geq 2$.
Then for any smooth $(\mathrm{n}-1)$*-form* α, $\int_\partial \alpha = \int_{\mathrm{M}} \mathrm{d}\alpha$.

Proof Idea: Deformation Invariance of Volume (cont.)

Pick an arbitrary point $s \in U$ and a real smooth connected curve $\gamma : [0, 1] \to U$ with $\gamma(0) = 0$ and $\gamma(1) = s$:

$$(K_{X'_s} + B'_s + \boldsymbol{\beta}_{X'_s})^m - (K_{X'_0} + B'_0 + \boldsymbol{\beta}_{X'_0})^m = \int_{M_V} d(K_{X'} + B' + \boldsymbol{\beta}_{X'})^m = 0.$$

Hence

$$t \longmapsto \mathrm{vol}(K_{X_t} + B_t + \boldsymbol{\beta}_{X_t})$$

is constant in the neighborhood $0 \in U$. □

Deformation Invariance of Volume and Plurigenera for Kähler Threefolds

In dimension ≤ 3, the full MMP is available by Höring–Peternell 16', Campana–Höring–Peternell 15', Das–Hacon 23', Das–Hacon–Păun 24', Das–Hacon 25', Das–Ou 23', etc.

Theorem (Hacon-L.-Rao 26')

Let $f : X \to S$ be a smooth family of compact Kähler threefolds over a smooth, connected, and relatively compact curve S. Then for any integer $m \geq 1$, the m-genus $P_m(X_t)$ is constant for all $t \in S$. Furthermore, assume that there exists a Kähler class ω_X on X such that $\omega_{X_t} := \omega_X|_{X_t}$. Then for every $\delta \geq 0$, the function

$$t \longmapsto \mathrm{vol}(K_{X_t} + \delta\,\omega_{X_t})$$

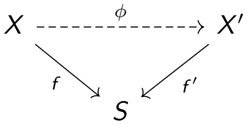
is independent of $t \in S$.

We prove the deformation invariance of plurigenera and omit the proof of invariance of volumes.

Proof Idea: Invariance of Plurigenera for Kähler Threefolds

Our proof follows the idea of Nakayama 85': Since the relative threefold MMP is known (cf. [DHP 24']), it terminates at either a relative minimal model or a relative Mori fiber space.

Case 1: The K_X-relative MMP terminates at a relative minimal model over S, say $X \dashrightarrow X'/S$, with $f' : X' \to S$.



(1) If the central fiber X_0 is contracted by ϕ, then $P_m(X_0) = 0$ for all $m \geq 1$. Since f is flat, the upper semicontinuity of cohomological dimension implies $P_m(X_t) = 0$ for all $t \in S$ and $m \geq 1$.

(2) If X_0 is not contracted by the MMP, then since it terminates at a relative minimal model, $K_{X'}$ is nef over S. By the Abundance conjecture, $K_{X'}$ is semi-ample over S.

Proof Idea: Invariance of Plurigenera for Kähler Threefolds (Cont.)

(3) By Kollár's torsion freeness theorem, $R^i f'_* \mathcal{O}_{X'}(mK_{X'})$ is locally free on S for all $i \geq 0$ and $m \geq 1$.
(4) By the Grauert base-change theorem, the plurigenera $P_m(X_t)$ are constant in the family for all $m \geq 1$.
Case 2: The K_X-relative MMP terminates at a relative Mori fiber space $\psi : X' \to Z$ over S, henc $P_m(X'_0) = 0$ for all $m \geq 1$, and consequently $P_m(X_0) = 0$ for all $m \geq 1$. By the upper semicontinuity of cohomological dimension, $P_m(X_t) = 0$ for all $t \in S$ and $m \geq 1$. $\square$

Thanks!