# Rational curves in the Kähler MMP and a Kähler criterion

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# Kähler varieties

**Definition (Kähler varieties)**

*Let $X$ be a normal complex variety. A Kähler form on $X$ is a closed real $(1,1)$-form $\omega$ which is locally of the form $i\partial\bar{\partial}\varphi$ with $\varphi$ smooth and strictly plurisubharmonic. A Kähler variety is a complex variety with a Kähler form.*

**Definition (Fujiki varieties)**

*A compact complex variety $X$ is in Fujiki's class $\mathcal{C}$ if it is bimeromorphic to a compact Kähler manifold.*

Question: When a Fujiki variety is Kähler?

# Kähler surfaces

**Theorem (Kodaira, Miyaoka, Siu, Lamari, Buchdahl)**

*A compact complex surface is Kähler if and only if $b_1$ is even.*

**Theorem (Takayama [Tak03])**

*If $X$ has klt singularities (e.g. $X$ smooth) and $\widetilde{X} \to X$ is a resolution, then $\pi_1(\widetilde{X}) \cong \pi_1(X)$. In particular $\pi_1$, and hence $b_1$, is a bimeromorphic invariant.*

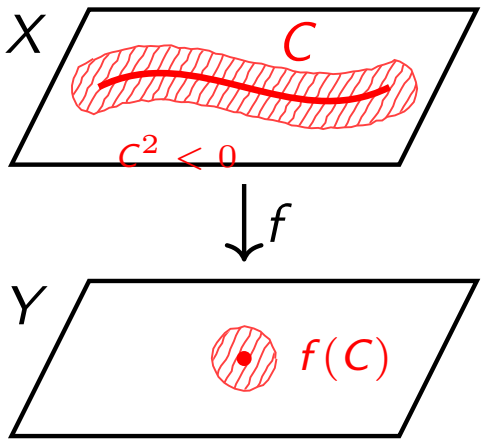
Hence being Kähler is a bimeromorphic invariant for complex surfaces.

# Projective surfaces

Let $X$ be a smooth projective surface and $C \subset X$ a curve with $C^2 < 0$. Since $D \cdot C \geq 0$ for every curve $D \neq C$, the class of $C$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$.

**Theorem (Grauert [Gra62])**

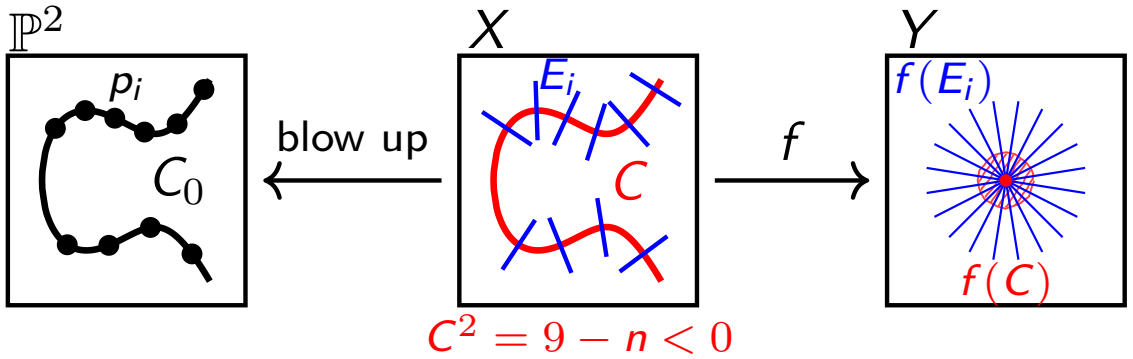
*Let $C = \bigcup_i C_i$ be a compact connected curve on a smooth complex surface $X$. There is a proper bimeromorphic morphism $f : X \to Y$ onto a normal surface contracting exactly $C$ to a point if and only if the matrix $(C_i \cdot C_j)$ is negative definite.*




In general $Y$ is only a normal compact complex surface. Is $Y$ projective?

# Projectivity of the contraction: two cases

**Case 1:** $C$ **rational.** If $C \cong \mathbb{P}^1$, then $Y$ is projective (Artin [Art62]).
**Case 2:** $C$ **elliptic.** Then $Y$ need not be projective.




If $Y$ is projective with $H$ ample, then $f^*H$ is a big line bundle on $X$, trivial on $C$. Whether such a line bundle exists depends on $C$.

## Example

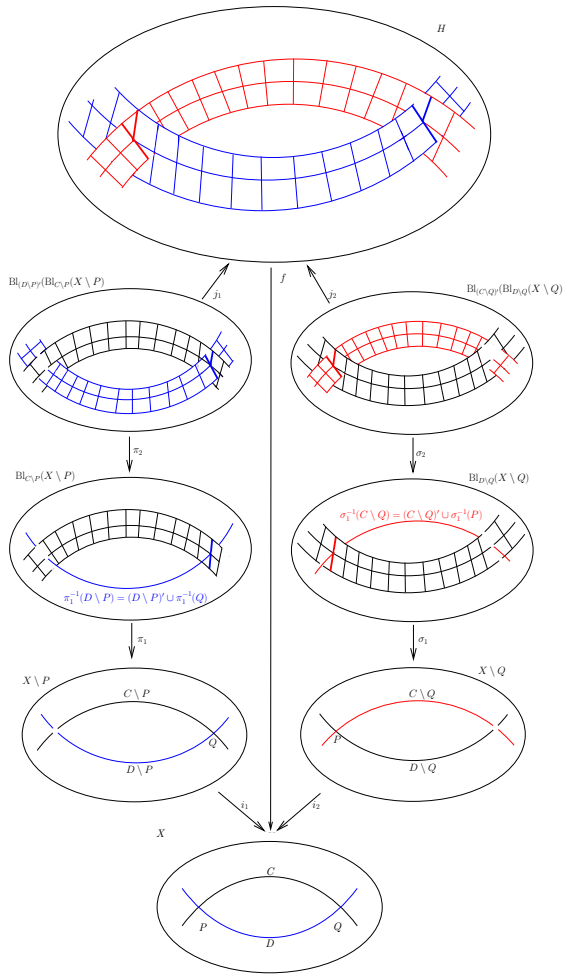
*Let $X$ be $\mathbb{P}^2$ blown up at $n \geq 10$ very general points $p_i$ of a smooth cubic $C_0$, and $C$ its strict transform, so $C^2 = 9 - n < 0$. A line bundle $L = aH - \sum b_i E_i$ is trivial on $C$ only if $a\,H|_C \sim \sum b_i p_i$; very general points satisfy no such relation, so $L$ is trivial and $Y$ is not projective.*

# Hironaka's example

Let $C, D$ be smooth rational curves in a smooth projective threefold $X$, meeting transversally at two points $P, Q$.

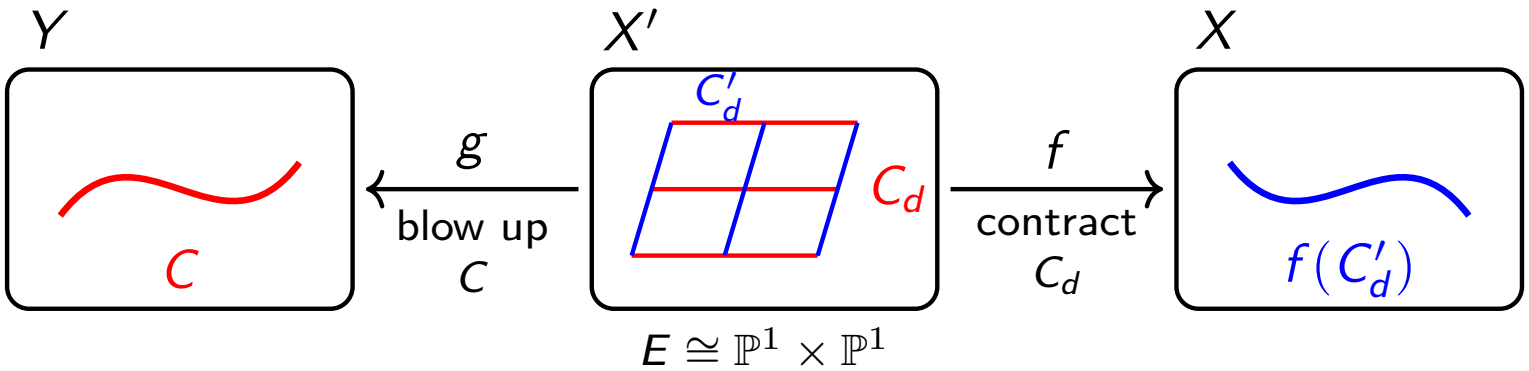
Near $P$, blow up $C$ and then $D$; near $Q$, blow up $D$ and then $C$. Gluing gives a smooth compact Moishezon threefold $H \to X$, the *Hironaka variety*; in particular $H$ is in Fujiki's class $\mathcal{C}$.

Over $P$ and $Q$ there are rational curves $L_P, L_Q$ with $L_P + L_Q \equiv 0$. Hence $H$ is not Kähler.
Note that $-[L_Q] \equiv [L_P] \in \overline{\mathrm{NA}}(H)$, the cone of positive currents (defined below); this is the obstruction to being Kähler.



# A non-Kähler example: flopping a curve on a quintic

Let $Y$ be a general projective quintic threefold and $C \subset Y$ a smooth rational curve of degree $d$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Blow up $C$, then contract the exceptional divisor $E$ along the other ruling; the result $X$ is a smooth Moishezon threefold which is not Kähler:




$X$ is not Kähler/projective. For a line $L \subset Y$ disjoint from $C$, we have

$$C_d \equiv C'_d + d\,L$$

on $X'$. As $f$ contracts $C_d$, $-[f(C'_d)] = d\,[f(L)] \in \overline{\mathrm{NA}}(X)$. A Kähler class $\omega$ would give $\omega \cdot f(C'_d) > 0$ and $-\omega \cdot f(C'_d) \geq 0$, impossible. So $X$ is not Kähler, hence not projective.

# Positivity of classes

Let $X$ be a compact normal variety in Fujiki's class $\mathcal{C}$, and fix a hermitian form $\omega$ on $X$. We work in the *Bott–Chern cohomology* $H^{1,1}_{\mathrm{BC}}(X)$.

- A class $\alpha$ is *Kähler* if it contains a Kähler form;
- A class $\alpha$ is *nef* if for every $\epsilon > 0$ it contains a form $\geq -\epsilon\omega$;
- A class $\alpha$ is *big* if it contains a Kähler current $T \geq \epsilon\omega$;
- A class $\alpha$ is *pseudo-effective* if it contains a closed positive current.

The *generalized Mori cone* $\overline{\mathrm{NA}}(X) \subset H^{1,1}_{\mathrm{BC}}(X)^*$ is the closed cone generated by the classes of closed positive currents of bidimension $(1,1)$.

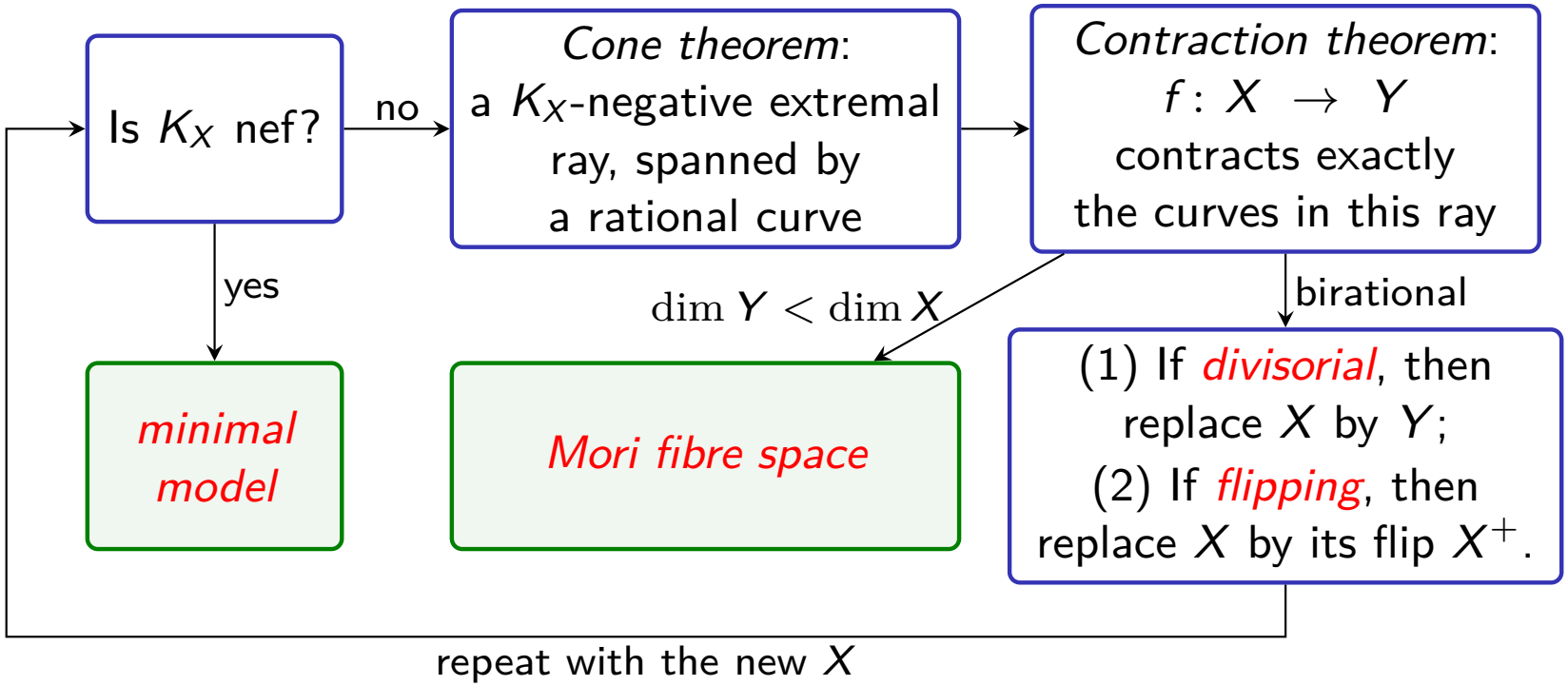
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# The minimal model program

Let $X$ be a projective/Kähler variety with klt singularities. The minimal model program goes as follows:




For klt pairs $(X, B)$ with $B$ big, the loop (with scaling) terminates [BCHM10].

# Rational curves and the cone theorem

**Theorem (Mori, Kawamata, Kollár, Shokurov)**

*Let $(X, B)$ be a projective klt pair. Then*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+B\geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i],$$

*with $C_i$ rational curves, $0 < -(K_X + B) \cdot C_i \leq 2 \dim X$.*

**Theorem (Hacon–Păun [HP24])**

*Let $(X, B + \beta)$ be a compact Kähler generalized klt pair, with $X$ strongly $\mathbb{Q}$-factorial. Then*

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X+B+\beta_X\geq 0} + \sum_i \mathbb{R}_{\geq 0}[\Gamma_i],$$

*with $\Gamma_i$ rational curves, $0 < -(K_X + B + \beta_X) \cdot \Gamma_i \leq 2 \dim X$.*

Here a *generalized pair* $(X, B + \beta)$ [BZ16, DHY23] is a boundary divisor $B$ together with a closed b-$(1,1)$-current $\beta$ which is nef on some higher model.

# The contraction theorem

**Theorem (Kawamata, Shokurov)**

*Let $(X, B)$ be a projective klt pair and $R \subset \overline{\mathrm{NE}}(X)$ a $(K_X + B)$-negative extremal ray. Then there is a contraction $f : X \to Y$ onto a normal projective variety which contracts exactly the curves with class in $R$.*

**Theorem (Hacon–Xie [HX26])**

*Let $(X, B + \beta)$ be a compact Kähler generalized klt pair, with $X$ strongly $\mathbb{Q}$-factorial, and $R \subset \overline{\mathrm{NA}}(X)$ a $(K_X + B + \beta_X)$-negative extremal ray. Then the contraction $f : X \to Y$ of $R$ exists and $f$ is projective.*

Whether $Y$ is again Kähler is the question we will answer in this talk.

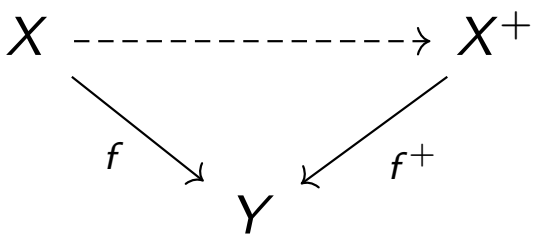
# Existence of flips

**Theorem (Hacon–McKernan [HM10])**

*Let $(X, B)$ be a projective $\mathbb{Q}$-factorial klt pair and $f : X \to Y$ a flipping contraction. Then the flip $X^+$ exists, and $X^+$ is projective.*

**Theorem (Das–Hacon [DH25])**

*Let $(X, B + \beta)$ be a compact Kähler generalized klt pair, with $X$ strongly $\mathbb{Q}$-factorial, and $f : X \to Y$ a flipping contraction. Then the flip $f^+ : X^+ \to Y$ exists, and $f^+$ is projective.*



# The Kähler minimal model program

**Theorem (Birkar–Cascini–Hacon–McKernan [BCHM10])**

*Let $(X, B)$ be a projective klt pair with $B$ big. If $K_X + B$ is pseudo-effective, it has a good minimal model; otherwise it has a Mori fibre space.*

**Conjecture (Kähler MMP [HX26])**

*Let $(X, B + \beta)$ be a compact Kähler generalized klt pair with $B + \beta_X$ modified big. If $K_X + B + \beta_X$ is pseudo-effective, it has a good minimal model; otherwise it has a Mori fibre space.*

Following BCHM, each step needs cone theorem [HP24], contraction theorem [HX26] and existence of flips [DH23, DH25], and:

**Question**

*Is the output of each flip and divisorial contraction again Kähler?*

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# Main results

**Theorem A (Kähler MMP preserves Kähler condition)**

*Let $(X, B + \beta)$ be a generalized klt pair with $X$ compact Kähler and strongly $\mathbb{Q}$-factorial, and $B + \beta_X$ big. Then every flip and every divisorial contraction of $X$ is again Kähler.*

**Theorem B (A Kählerness criterion)**

*Let $(X, B + \beta)$ be a generalized klt pair with $X$ compact in Fujiki's class $\mathcal{C}$ and strongly $\mathbb{Q}$-factorial. Then $X$ is Kähler if and only if there is no rational curve $C \subset X$ with $-[C] \in \overline{\mathrm{NA}}(X)$.*

# The key tool: a Kähler criterion for adjoint classes

**Theorem (Hacon–L.–Xie)**

*Let $(X, B+\beta)$ be a compact generalized klt pair with $X$ in Fujiki's class $\mathcal{C}$, and $\alpha = [K_X + B + \beta_X]$ nef, big (and NQC). If $\alpha$ is not Kähler, then there is a rational curve $C \subset X$ with $\alpha \cdot C = 0$.*

NQC: $\alpha = \sum_i a_i \alpha_i$ with $a_i > 0$ and $\alpha_i \in H^2(X, \mathbb{Q})$ non-negative on $\overline{\mathrm{NA}}(X)$ [HX26, Definition 2.43].

Idea of the proof. Suppose $\alpha$ is not Kähler. We proceed in three steps.

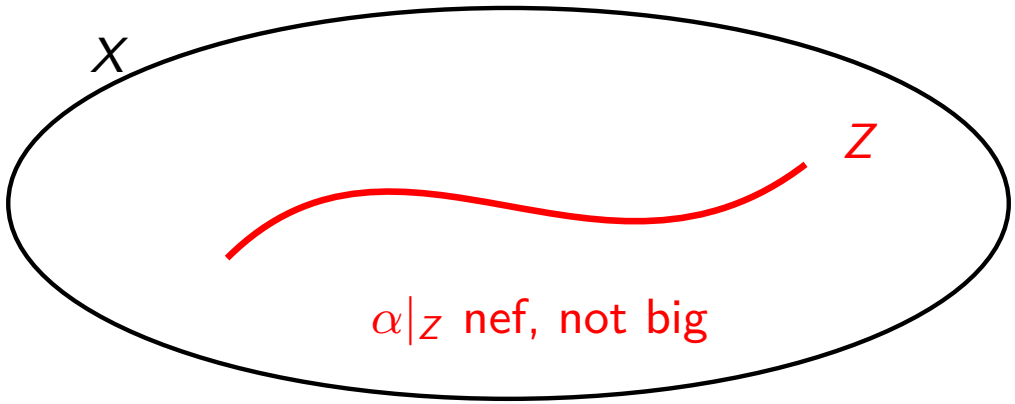
1. $\alpha$ is Kähler away from its null locus [CT15], so there is a component $Z \subset \mathrm{Null}(\alpha)$ on which $\alpha|_Z$ is not big.
2. We show that $Z$ is uniruled. Otherwise $K_Z$ would be pseudo-effective [Ou25], and then $\alpha|_Z$ would be big.
3. On a general fibre of the MRC fibration of $Z$, $\alpha$ is nef but not big; an MMP there produces an $\alpha$-trivial rational curve.

# Step 1: the null locus

The *null locus* of the nef and big class $\alpha$ is

$\mathrm{Null}(\alpha) = \bigcup\{V \subset X \text{ positive-dimensional subvarieties} : \int_V \alpha^{\dim V} = 0\}$.

If $\alpha$ is not Kähler, then $\mathrm{Null}(\alpha) \neq \emptyset$.




Let $Z$ be a component of $\mathrm{Null}(\alpha)$ of maximal dimension. Then $\alpha|_Z$ is nef but not big, and we look for the rational curve inside $Z$.

# Step 2a: adjunction to a divisor over $Z$

$Z$ may have higher codimension (not codimension 1), so we cannot do adjunction to $Z$ itself. Instead we replace $Z$ by a divisor $S$ lying over it.

- *Make $Z$ an lc centre.* Take a Kähler current $\eta \in \alpha$ singular exactly along $\mathrm{Null}(\alpha)$ [CT15]; adding a suitable multiple $c\,\eta$ to the boundary makes $Z$ an lc centre, with a unique lc place: a divisor $S$ on a resolution of $X$. The new adjoint class is $(1+c)\,\alpha$.
- *Adjunction on $S$.* Restricting this class to the divisor $S$ gives an adjoint class on $S$, pulled back from a resolution $Z'$ of $Z$:

$$\begin{array}{ccc} S & \subset & X' \\ {\scriptstyle f}\downarrow & & \downarrow \\ Z' & \longrightarrow & Z \subset X \end{array} \qquad K_S + \Delta_S + \boldsymbol{\delta}_S \equiv (1+c)\, f^* \alpha_{Z'},$$

where $\alpha_{Z'}$ is the pullback of $\alpha|_Z$ and $\boldsymbol{\delta}_S$ is Kähler.

# Step 2b: $Z$ is uniruled

Suppose $Z$ is not uniruled; we show that $\alpha|_Z$ is big, a contradiction. Writing $K_S = K_{S/Z'} + f^*K_{Z'}$ in the adjunction formula, for a small Kähler class $\omega_{Z'}$,

$$(1+c)\, f^*\alpha_{Z'} - f^*\omega_{Z'} \equiv \underbrace{f^*K_{Z'}}_{\text{psef by [Ou25]}} + \underbrace{\left(K_{S/Z'} + \Delta_S + \boldsymbol{\delta}_S - f^*\omega_{Z'}\right)}_{\text{relative part: psef?}}.$$

If the relative part is pseudo-effective, then so is $(1+c)\,\alpha_{Z'} - \omega_{Z'}$, and $\alpha_{Z'}$ is big. The relative part is controlled by:

**Theorem (Hacon–Păun [HP24, Theorem 2.2])**

*Let $g\colon V \to W$ be a projective fibration of compact Kähler varieties and $(V, B+\boldsymbol{\beta})$ generalized lc over general points of $W$. If*

$$K_V + B + \boldsymbol{\beta}_V \equiv g^*\gamma + L \quad \text{with } L \geq 0,$$

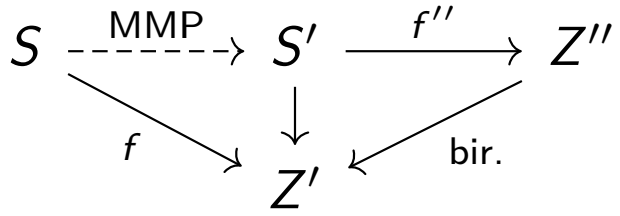
*then $K_{V/W} + B + \boldsymbol{\beta}_V$ is pseudo-effective.*

Here $W$ is smooth, and $L$ is an effective $\mathbb{R}$-Cartier divisor or a $\mathbb{Q}$-line bundle with sections on the general fibre.

# Step 2c: the relative MMP

Hacon–Păun needs a *boundary* whose adjoint class is a *pullback* from the base.

- $K_S + \Delta_S + \boldsymbol{\delta}_S$ is a pullback, but $\Delta_S$ may have coefficients $\geq 1$ or $< 0$.
- Truncate the boundary, $\tilde{\Delta}_S = \Delta_S - \Delta_S^{\geq 1} + \Delta_S^{<0}$, and run the MMP of $K_S + \tilde{\Delta}_S + \boldsymbol{\delta}_S$ over $Z'$ to make it a pullback again:




$$K_{S'} + \tilde{\Delta}_{S'} + \boldsymbol{\delta}_{S'} = (f'')^* \lambda'',$$

where $S'$ is a good minimal model, $f''$ its lc model and $\lambda'' \in H^{1,1}_{\mathrm{BC}}(Z'')$ is Kähler over $Z'$.

Hacon–Păun applies on $S'$, so the relative part is pseudo-effective and $\alpha_{Z'}$ is big, a contradiction. So $Z$ is uniruled.

# Step 3: finding the rational curve

Let $\varphi\colon Z' \dashrightarrow W$ be the MRC fibration, with general positive-dimensional fibre $F$; it is projective.

$$\begin{array}{ccccc} Z' & \xrightarrow{\varphi} & W & & \text{not uniruled} \\ \uparrow & & & & \\ F & \xrightarrow{\text{MMP}} & F_m & \xrightarrow{g} & T \end{array}$$

rationally connected

- We first show that $\alpha_F = \alpha_{Z'}|_F$ is nef, not big. Otherwise, as $W$ is not uniruled, the argument of the previous step makes $\alpha_{Z'}$ big.
- For $t \gg 0$, the $(K_F + t\,\alpha_F + \omega_F)$-MMP ends with a Mori fibre space $g\colon F_m \to T$.
- By the NQC condition, the MMP is $\alpha_F$-trivial and $F_m$ is covered by $\alpha_{F_m}$-trivial rational curves, therefore $F$ is covered by $\alpha_F$-trivial rational curves.

□

# Theorem A: the Kähler MMP stays Kähler

**Theorem A (Hacon–L.–Xie)**

*Let $(X, B+\beta)$ be a generalized klt pair, with $X$ compact Kähler and strongly $\mathbb{Q}$-factorial and $B+\beta_X$ big. If $f : X \to Z$ is a flipping or divisorial contraction, then $Z$ is Kähler, and so is the flip $X^+$.*

Theorem A is a corollary of the Kähler criterion, applied on $Z$.

1. $f$ contracts an extremal ray $R$, cut out by a nef class $\alpha = [K_X + B + \beta_X + \omega]$, $\omega$ Kähler: $\alpha \cdot C = 0 \iff [C] \in R$. It is big, since $f$ is birational.
2. Descend $\alpha$ to $Z$: $\alpha = f^*\alpha_Z$, where $\alpha_Z = [K_Z + B_Z + \beta_Z + f_*\omega]$ is again an adjoint class on $Z$, nef and big.
3. A curve $C \subset Z$ is the image of a curve $C' \subset X$ not in $R$, so $\alpha_Z \cdot C = \alpha \cdot C' > 0$. With no $\alpha_Z$-trivial curve, $\alpha_Z$ is Kähler.
4. *The flip.* $X^+ \to Z$ is projective and $Z$ is Kähler, so $X^+$ is Kähler. □

# Theorem B: when is a Fujiki variety Kähler?

**Theorem B (Hacon–L.–Xie)**

*Let $(X, B+\beta)$ be a generalized klt pair with $X$ compact in Fujiki's class $\mathcal{C}$. If $X$ is not Kähler, then either*

1. *$X$ contains a rational curve $C$ with $-[C] \in \overline{\mathrm{NA}}(X)$, or*
2. *there is a small $\mathbb{Q}$-factorial Kähler modification $X^{\mathrm{qf}} \to X$.*

**Corollary**

*If moreover $X$ is strongly $\mathbb{Q}$-factorial, then $X$ is Kähler if and only if there is no rational curve $C$ with $-[C] \in \overline{\mathrm{NA}}(X)$.*

Theorem B follows from Theorem A: an MMP over $X$ from a Kähler resolution stays Kähler, and ends in (2) or produces the curve in (1).

Thank you for your attention!

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