Nakayama's Invariance of Plurigenera Summer 2025

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The aim of this note is to summarize Nakayama's results on the invariance of plurigenera. We will first briefly review the technical core of the proof, stating several extension theorems without giving their proofs. The story deserves another note. We then summarize Nakayama's results on the deformation of plurigenera. The major references are [Nak86, Nak87, Nak04].

Contents

1 Review of Nakayama's extension theorems

1.1 Extension when nef and big condition holds

1.2 Extension when nef and abundant condition holds

2 Lower semi-continuity of numerical Kodaira dimension

Theorem 1. Let $f : \mathcal{X} \to S$ be a projective morphism from a normal complex analytic variety to the smooth curve. Assume that the special fiber has the irreducible decomposition $\mathcal{X}_0 = \bigcup \Gamma_i$.

The numerical Kodaira dimension κ_σ is lower semi-continuous in the sense that, for a general fiber $\mathcal{X}_s$,

$$\kappa_\sigma(\mathcal{X}_s) \geq \max \kappa_\sigma(\Gamma_i).$$

As a corollary, we can show the invariance of numerical Kodaira dimension when fibers admits canonical singularities.

Corollary 2. Let S be an algebraic variety, let $\mathcal{X}$ be a complex veriety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities for any $t \in S$.

(1) Then the numerical Kodaira dimension $\nu(X_t)$ is constant on $t \in S$.
(2) In particular, if one fiber X_0 is of general type, then so are all the fibers.

3 Lower semi-continuity of plurigenera when abundance on generic fibers

Theorem 3 ([Nak04, Theorem V.4.5])**.** Consider the projective morphism $\mathcal{X} \to S$ with connected fibers, from a normal complex analytic space onto non-singular curve.

Suppose that $\kappa(\mathcal{X}_s) = \kappa_\sigma(\mathcal{X}_s)$ for a 'general' fiber $\mathcal{X}_s$. Let X be the set of indices i with $\kappa_\sigma(\Gamma_i) = \kappa(\mathcal{X})$. Then, for any $m > 0$

$$P_m(\mathcal{X}_s) \geq \sum_{i \in I} P_m(\Gamma_i).$$

4 Lower semi-continuity of plurigenera count general type components on central fiber

Theorem 4 ([Nak04, Theorem 4.3])**.** Consider the projective morphism $\mathcal{X} \to S$ with connected fibers, from a normal complex analytic space onto non-singular curve. Let I be the set of indices i such that Γ_i is of general type. If $I \neq \emptyset$, then, for any $m > 0$,

$$P_m(\mathcal{X}_s) \geq \sum_{i \in I} P_m(\Gamma_i).$$

5 Minimal model conjecture + Abundance conjecture implies lower semi-continuity of plurigenera

Theorem 5. Let $f : X \to C$ be a semistable projective morphism from non-singular variety X onto a non-singular curve C. Assume the minimal model conjecture and abundance conjecture are true.

Then the following inequality for plurigenera holds

$$\sum_{i \in I} P_m(\Gamma_i) \leq P_m(X_t)$$

for general fiber X_t.

References

[Nak86] N. Nakayama, *Invariance of the plurigenera of algebraic varieties under minimal model conjectures*, Topology **25** (1986), no. 2, 237–251.

[Nak87] ______, *The lower semicontinuity of the plurigenera of complex varieties*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 551–590.

[Nak04] ______, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.