Hodge Theory

Complete Lecture Notes

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Responsibility Statement

These notes are based on lectures given by Prof. Kang Zuo. I have prepared and edited them, and I am responsible for possible errors, inaccuracies, or omissions that may remain.

Contents

Lecture 1: June 5, 2026

One of the main goals of this course is to use Hodge theory to understand the global structure of moduli spaces.

We begin with a very simple example. Today's lecture has two main topics: (1) the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the simplest hyperbolic Riemann surface, and (2) the Hodge structure on elliptic curves.

1.1 Geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Let us consider the parameter space

$$\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Our story starts from the following Picard–Fuchs equation:

$$\lambda(\lambda - 1)\pi'' + (2\lambda - 1)\pi' + \frac{1}{4}\pi = 0.$$

This ordinary differential equation has regular singularities at $\{0, 1, \infty\}$, and outside these points its solution space is a two-dimensional vector space. A local basis of solutions is given by

$$A(\lambda) = \int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \qquad B(\lambda) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}},$$

where γ and δ are the loops on the Riemann sphere shown below.

These functions are called the *periods* of the Legendre family. On any disk

$$\Delta \subset \mathbb{P}^1 \setminus \{0, 1, \infty\},$$

the periods are single-valued holomorphic functions of λ. However, if we analytically continue them along a loop α around one of the points in $\{0, 1, \infty\}$ and then return to the original value of λ, the resulting solutions may differ from the original ones:

$$(A', B') = T_\alpha(A, B),$$

where T_α is called the monodromy transformation. It turns out that

$$T_\alpha \in SL_2(\mathbb{Z}).$$

We then define the period map by

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \longrightarrow \mathbb{C}, \qquad \lambda \longmapsto \frac{B(\lambda)}{A(\lambda)}.$$

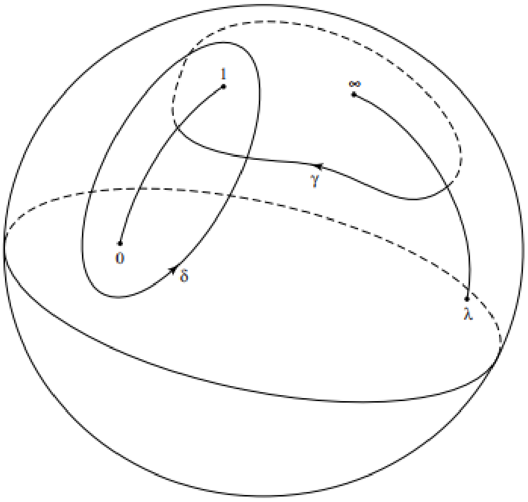

Figure 1.1: Cycles used to define the periods of the Legendre family.

One checks that the image of this period map lies in the upper half-plane

$$\mathscr{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

and that the map is holomorphic. Passing to the universal cover, we obtain a holomorphic map

$$\widetilde{\tau} : \mathbb{P}^1 \setminus \widetilde{\{0, 1, \infty\}} \longrightarrow \mathscr{H}.$$

One can verify, for example by following the original approach of Abel and Jacobi and computing the derivative of $\widetilde{\tau}$, that $\widetilde{\tau}$ is a biholomorphism onto $\mathscr{H}$.

It is worth mentioning that Deligne and Mostow considered an analogous problem in much greater generality, studying complements of hyperplane arrangements in projective spaces.

The biholomorphism $\widetilde{\tau}$ is equivariant: the domain carries the natural action of

$$\pi_1\Big(\mathbb{P}^1 \setminus \{0, 1, \infty\}\Big),$$

while on $\mathscr{H}$ we have the action induced by the monodromy representation

$$\rho : \pi_1\Big(\mathbb{P}^1 \setminus \{0, 1, \infty\}\Big) \longrightarrow \operatorname{Aut}(V_\lambda), \qquad \alpha \longmapsto T_\alpha,$$

where V_λ is the two-dimensional solution space. The image satisfies

$$\rho\Big(\pi_1\Big(\mathbb{P}^1 \setminus \{0, 1, \infty\}\Big)\Big) = \Gamma(2) = \ker(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{F}_2)),$$

which is a proper subgroup of $SL_2(\mathbb{Z})$. Descending the equivariant biholomorphism $\widetilde{\tau}$, we obtain

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \cong \mathscr{H}/\Gamma(2).$$

Many moduli spaces carry a natural Hodge theory, but some do not.

1.2 Hodge theory of elliptic curves

We now consider the Legendre family

$$V = \left\{ y^2 = x(x-1)(x-\lambda) \mid \lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\} \right\}.$$

Its fiber over $\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ is denoted by E_λ^o. Thus the family endows the hyperbolic space

$$\mathbb{P}^1 \setminus \{0,1,\infty\}$$

with a natural "moduli space" interpretation. Later, we will compactify this family and define it over $\mathbb{P}^1$. We denote the corresponding compactification of E_λ^o by E_λ. This compactification introduces degenerations of Hodge structures. We will also introduce the notion of *degree*; using degree, we can see why the derivative of the period map is an isomorphism.

We first discuss the **topology** of E_λ. For the projective curve E_λ, choosing a symplectic basis δ, γ gives a decomposition

$$H_1(E_\lambda, \mathbb{Z}) = \mathbb{Z}\delta \oplus \mathbb{Z}\gamma.$$

The intersection form with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This fiberwise intersection form gives the polarization of the Hodge structure and is closely related to the Hodge metric in families. The Hodge metric plays an important role in variation of Hodge structures.

Hodge theory combines three kinds of geometry: topology, differential geometry, and complex geometry. We next discuss the complex geometry of E_λ. Regarded as a Riemann surface, E_λ carries the holomorphic differential form

$$\omega_\lambda = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

Although this expression is written in one affine chart, it glues with the corresponding expressions in the other affine charts, and hence defines a global holomorphic differential form. When performing loop integration, this affine expression is sufficient, but it is useful to remember that the form is globally defined. By the basic theory of elliptic curves, the space of holomorphic 1-forms is one-dimensional, so ω_λ is unique up to scaling.

Now consider periods. By de Rham theory, a closed 1-form defines a cohomology class, equivalently a functional on homology:

$$\int_{-} \omega_\lambda : H_1(E_\lambda, \mathbb{C}) \longrightarrow \mathbb{C}, \qquad \alpha \longmapsto \int_\alpha \omega_\lambda.$$

With respect to the dual basis δ^*, γ^* of $H^1(E_\lambda, \mathbb{C})$, we have

$$[\omega_\lambda] = \delta^* \int_\delta \omega_\lambda + \gamma^* \int_\gamma \omega_\lambda.$$

We call

$$(A(\lambda), B(\lambda)) = \left(\int_\delta \omega_\lambda, \int_\gamma \omega_\lambda \right)$$

the periods of E_λ. These periods depend on the choice of ω_λ. However, as noted above, the holomorphic 1-form is unique up to scaling. Therefore the ratio

$$\tau(\lambda) = \frac{B(\lambda)}{A(\lambda)}$$

is independent of this scaling, provided $A(\lambda) \neq 0$. The fact that $A(\lambda)$ is nonzero follows from the following theorem.

Theorem 1.1. *Let $H^{1,0}_\lambda \subset H^1(E_\lambda, \mathbb{C})$ be the subspace spanned by $[\omega_\lambda]$. Then*

$$H^1(E_\lambda, \mathbb{C}) = H^{1,0}_\lambda \oplus H^{0,1}_\lambda.$$

Here the left-hand side is topological, while the decomposition on the right-hand side depends on the complex structure of E_λ.

Proof. Consider the intersection pairing

$$H^1(E_\lambda, \mathbb{C}) \times H^1(E_\lambda, \mathbb{C}) \longrightarrow \mathbb{C},$$

which is induced by the intersection form on $H_1(E_\lambda, \mathbb{C})$. If

$$A = \int_\delta \omega_\lambda, \qquad B = \int_\gamma \omega_\lambda,$$

then, up to the convention for the chosen symplectic basis,

$$\mathrm{i} \int_{E_\lambda} [\omega] \cup [\overline{\omega}] = 2\ \mathrm{Im}(B\overline{A}).$$

On the other hand, locally we can write $\omega = f\,dz$. Hence

$$\mathrm{i}\,\omega \wedge \overline{\omega} = \mathrm{i}|f|^2\,dz \wedge d\overline{z} = 2|f|^2\,dx \wedge dy,$$

where $dx \wedge dy$ is the orientation defined by the complex structure. Thus the integrand is a positive function times the volume element. Therefore

$$\mathrm{i} \int_{E_\lambda} \omega \wedge \overline{\omega} > 0,$$

and consequently

$$\mathrm{Im}(B\overline{A}) > 0.$$

In particular, $A \neq 0$ and $B \neq 0$.

We may therefore normalize ω_λ so that $A = 1$. Under this normalization,

$$\mathrm{Im}\,B > 0.$$

Suppose now that $H^{1,0}_\lambda$ and $H^{0,1}_\lambda$ do not give a direct-sum decomposition of $H^1(E_\lambda, \mathbb{C})$. Since both are one-dimensional subspaces, this would imply

$$H^{1,0}_\lambda = H^{0,1}_\lambda.$$

Hence $[\overline{\omega}] = c[\omega]$ for some $c \in \mathbb{C}$. Using the normalization $A = 1$, this gives

$$\delta^* + \overline{B}\gamma^* = c(\delta^* + B\gamma^*).$$

Comparing coefficients, we obtain $c = 1$ and $\overline{B} = B$, contradicting $\mathrm{Im}\,B > 0$. Therefore

$$H^1(E_\lambda, \mathbb{C}) = H^{1,0}_\lambda \oplus H^{0,1}_\lambda.$$

□

Locally on a disk

$$\Delta \subset \mathbb{P}^1 \setminus \{0, 1, \infty\},$$

the Legendre family is a smooth product family by Ehresmann's theorem. The period map

$$\Delta \longrightarrow \mathscr{H}, \qquad \lambda \longmapsto B(\lambda),$$

after the normalization $A(\lambda) = 1$, measures the variation of the complex structure, and it is holomorphic.

The definitions of the period functions A and B on a disk Δ depend on the choice of a symplectic homology basis $\{\delta, \gamma\}$. A different symplectic basis $\{\delta', \gamma'\}$ is related to the old one by

$$\begin{aligned} \delta' &= a\delta + b\gamma, \\ \gamma' &= c\delta + d\gamma, \end{aligned}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The periods with respect to the new basis are related to those with respect to the old basis by

$$\begin{aligned} A' &= aA + bB, \\ B' &= cA + dB. \end{aligned}$$

Thus the new period vector (A', B') is obtained by multiplying the old period vector (A, B) by the matrix T. Therefore the corresponding τ-invariants are related by the fractional linear transformation

$$\tau' = \frac{B'}{A'} = \frac{c + d\tau}{a + b\tau}.$$

We can quotient out this ambiguity by the $SL_2(\mathbb{Z})$-action, obtaining a well-defined holomorphic map

$$\Delta \longrightarrow \mathscr{H}/SL_2(\mathbb{Z}).$$

1.3 Non-constancy of the period map τ

We end this lecture by showing that the period map is nonconstant. From the moduli-space point of view this is clear: the elliptic curves vary in the family, and hence the corresponding moduli map is nonconstant. Nevertheless, we can also prove this more directly.

We show that τ is a nonconstant function of λ by studying its behavior along the ray $\lambda > 2$ on the real axis. More precisely, we will see that $\tau(\lambda)$ is asymptotically proportional to $\log \lambda$. Assume $\lambda \gg 2$. Then

$$\int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \sim \int_\delta \frac{dx}{x\sqrt{-\lambda}} = \frac{2\pi}{\sqrt{\lambda}}.$$

By deforming the path of integration, we find that

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = -2\int_1^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

The difference between the last integrand and

$$\frac{1}{x\sqrt{x-\lambda}}$$

has an asymptotically negligible integral. Thus the main contribution is the residual integral

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}}.$$

This integral can be computed explicitly:

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}} = \frac{4}{\sqrt{\lambda}}\arctan\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}} \sim \frac{2\mathrm{i}}{\sqrt{\lambda}}\log\lambda.$$

Consequently,

$$\tau(\lambda) \sim \frac{\mathrm{i}}{\pi}\log\lambda.$$

In particular, $\tau(\lambda)$ is nonconstant.

Lecture 2: June 8, 2026

We study the uniformization of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ by the period map.

In the previous lecture we defined the global period map

$$\tau : \mathbb{P}^1 \setminus \{0,1,\infty\} \longrightarrow \mathscr{H}/\mathrm{SL}_2(\mathbb{Z}), \qquad \lambda \longmapsto \frac{B(\lambda)}{A(\lambda)},$$

and proved that it is nonconstant. We now state the stronger uniformization result.

Theorem 2.1. *The period map*

$$\mathbb{P}^1 \setminus \widetilde{\{0,1,\infty\}} \longrightarrow \mathscr{H}$$

is an isomorphism. The following statements describe the uniformization supplied by the Legendre family:

(1) There is an isomorphism

$$\mathscr{H}/\Gamma(2) \simeq \mathbb{P}^1 \setminus \{0,1,\infty\}.$$

(2) The universal monodromy representation

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \longrightarrow \mathrm{SL}_2(\mathbb{Z})$$

is injective, and

$$\rho\left(\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})\right) = \Gamma(2).$$

Therefore

$$\Gamma(2) = \ker\left(\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})\right).$$

Hence

$$\mathbb{P}^1 \setminus \{0,1,\infty\} \simeq \mathscr{H}/\Gamma(2).$$

Proof. Consider the Legendre family

$$E = \{y^2 = x(x-1)(x-\lambda)\} \to \mathbb{P}^1 \setminus \{0,1,\infty\}.$$

We compactify the algebraic surface E to a fibration $\bar{E} \to \mathbb{P}^1$. The compactification has singular fibers over $\{0,1,\infty\}$; after a suitable double cover, all boundary monodromies become unipotent and the resulting family is semistable. We can do the semistable reduction via finite base change

$$\sigma : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z^2,$$

which is a $2:1$ map ramified at $0, \infty$. After this base change,

$$E_\sigma = E \times_\sigma \mathbb{P}^1$$

with 4 singular fibers at $S : \{0, +1, -1, \infty\}$. We may therefore use the standard theory of variations of Hodge structure.

We now define the local system

$$\mathbb{V}_{\mathbb{P}^1 \setminus S} := R^1 g_* \mathbb{Q}_{\bar{E}_\sigma \setminus \Delta}.$$

It is a local system whose transition functions are locally constant. We can then define the de Rham bundle

$$\mathcal{V}_{\mathbb{P}^1 \setminus S} := \mathbb{V}_{\mathbb{P}^1 \setminus S} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1 \setminus S}.$$

Locally we can write

$$\mathcal{V}|_\Delta = \{a\delta^* + b\gamma^* \mid a, b \in \mathcal{O}(\Delta)\}$$

and

$$\mathcal{V}|_{\Delta'} = \{a_1\delta^* + b_1\gamma^* \mid a_1, b_1 \in \mathcal{O}(\Delta)\}$$

Then Gauss–Manin connection is simply the derivative

$$\nabla(a\delta^* + b\gamma^*) = a'\delta^* + b'\gamma^*.$$

These local descriptions glue because the transition functions of the local system are constant.

We now form Deligne's canonical extension of of the Gauss–Manin connection $(\mathcal{V}_{\mathbb{P}^1 \setminus S}, \nabla_{\mathbb{P}^1 \setminus S})$. It extends to a logarithmic de Rham bundle $(\mathcal{V}_{\mathbb{P}^1}, \nabla_{\mathbb{P}^1})$ where connection with log pole

$$\nabla_{\mathbb{P}^1} : \mathcal{V}_{\mathbb{P}^1} \to \mathcal{V}_{\mathbb{P}^1} \otimes \Omega^1(\log S).$$

A logarithmic extension is not unique in general. In the present unipotent case, the canonical extension is characterized by nilpotent residue (equivalently, all residue eigenvalues are zero).

Now we consider the Hodge filtration bundle

$$\bigcup_{\lambda \in \mathbb{P}^1 \setminus S} \mathbb{C} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = E^{1,0}_\lambda \subset \mathcal{V}.$$

This is a holomorphic line bundle, and it extends across the boundary inside Deligne's canonical extension. We thus obtain the data of a polarized variation of Hodge structure

$$(\mathcal{V}_{\mathbb{P}^1}, \nabla_{\mathbb{P}^1}, E^{1,0}_{\mathbb{P}^1}, \text{intersection form}).$$

From Simpson's perspective, the same structure is related to solutions of the Hitchin–Simpson equations.

Finally, we introduce the log Higgs bundle, by taking associated graded. Recall that we have the Deligne's canonical extension

$$\nabla_{\mathbb{P}^1} : \mathcal{V}_{\mathbb{P}^1} \to \mathcal{V}_{\mathbb{P}^1} \otimes \Omega^1(\log S),$$

the Hodge line $E^{1,0}_{\mathbb{P}^1}$ is generally not flat for this connection. The Higgs field measures its failure to be flat, which is defined to be

$$\theta : E^{1,0}_{\mathbb{P}^1} \to \mathcal{V}_{\mathbb{P}^1}/E^{1,0}_{\mathbb{P}^1} \otimes \Omega^1_{\mathbb{P}^1},$$

Thus $\theta = 0$ if and only if $E^{1,0}_{\mathbb{P}^1}$ is flat. We now construct a graded Higgs bundle

$$(E, \theta) = (E^{1,0} \oplus \mathcal{V}/E^{1,0}, \theta)$$

In the next chapter we prove that $\theta \neq 0$ and that the derivative of the period map is an isomorphism.

□

Lecture 3: June 11, 2026

We continue with the basic example of the Legendre family of elliptic curves

$$E_\lambda := \{y^2 = x(x-1)(x-\lambda)\} \longrightarrow \mathbb{P}^1 \setminus \{0,1,\infty\}.$$

The main ideas in this lecture already appear in this example.

Recall from the previous lecture that the Legendre family admits a compactification over $\mathbb{P}^1$ with singular fibres over $\{0,1,\infty\}$. The fibres over 0 and ∞ are semistable, while the fibre over 1 is non-reduced. To avoid parabolic structures, we pass to a semistable reduction by the double cover

$$\mathbb{P}^1 \setminus \{0,+1,-1,\infty\} \longrightarrow \mathbb{P}^1 \setminus \{0,1,\infty\}, \qquad t \longmapsto \lambda = t^2.$$

Set

$$S := \{0,+1,-1,\infty\}.$$

After this base change, we obtain a family

$$g : E_\delta \longrightarrow \mathbb{P}^1 \setminus S$$

fitting into the commutative diagram

$$\begin{tikzcd}
E_\delta \arrow[r] \arrow[d, "g"'] & E \arrow[d] \\
\mathbb{P}^1 \setminus S \arrow[r] & \mathbb{P}^1 \setminus \{0,1,\infty\}.
\end{tikzcd}$$

The family g has a semistable compactification

$$\bar{g} : \bar{E}_\delta \longrightarrow \mathbb{P}^1$$

whose singular fibres lie over S. Although one can work directly with the original non-semistable family, doing so requires keeping track of the associated parabolic structure. The semistable reduction is therefore a convenient simplification.

3.1 The local system and the Hodge bundle on $\mathbb{P}^1 \setminus S$

Consider the weight-one local system

$$\mathbb{V}_{\mathbb{P}^1\setminus S} := R^1 g_* \mathbb{Q}_{E_\delta}$$

on $\mathbb{P}^1 \setminus S$. The associated flat holomorphic bundle is

$$(\mathcal{V}_{\mathbb{P}^1\setminus S}, \nabla^{\mathrm{GM}}) := (\mathbb{V}_{\mathbb{P}^1\setminus S} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^1\setminus S}, \nabla^{\mathrm{GM}}),$$

where ∇^{GM} is the Gauss–Manin connection. Locally, the transition functions of the local system are constant, and this is precisely what gives the flat connection.

This flat bundle carries the Hodge filtration. In the present family, the Hodge line bundle is locally generated by the holomorphic differential

$$\omega_t := \frac{dx}{\sqrt{x(x-1)(x-t^2)}}.$$

Thus

$$E^{1,0}_{\mathbb{P}^1\setminus S} := \bigcup_{t\in\mathbb{P}^1\setminus S} \mathbb{C}\omega_t \subset \mathcal{V}_{\mathbb{P}^1\setminus S}$$

is a holomorphic line subbundle. The corresponding period map is described by

$$\tau(t) = \mathbb{C}\omega_t \subset \mathcal{V}_t.$$

In other words, $\tau(t)$ records the Hodge line $H^{1,0}(E_t) \subset H^1(E_t, \mathbb{C})$.

3.2 Deligne's canonical extension over a punctured disk

To use algebraic geometry on a compact base, we must extend the flat bundle across the boundary. This is the role of Deligne's canonical extension.

Let Δ be the unit disk and let $\Delta^* = \Delta \setminus \{0\}$. Let $\mathbb{L}$ be a local system on Δ^*. The associated flat bundle is

$$(\mathcal{L}, \nabla) = (\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{\Delta^*}, \nabla).$$

We want to extend this bundle over Δ in a canonical way.

Let T be the local monodromy operator around 0. More explicitly, if $\gamma \in \pi_1(\Delta^*)$ is the positive generator and ℓ is a flat multivalued section, then analytic continuation of ℓ along γ gives $T(\ell)$. If the monodromy is unipotent, then

$$M := \log T = (T - I) - \frac{(T-I)^2}{2} + \frac{(T-I)^3}{3} - \cdots$$

is a finite sum. In this unipotent case, no choice of branch is involved in defining $\log T$. For general quasi-unipotent monodromy, Deligne's extension is obtained by fixing the eigenvalues of the residue in a prescribed interval.

Define the residue operator

$$R := -\frac{1}{2\pi i} M.$$

For a flat multivalued section ℓ, set

$$\widetilde{\ell} := \exp(R \log z)\ell.$$

Although ℓ and $\log z$ are individually multivalued, their combination is single-valued. Indeed, after going once around the origin,

$$\log z \longmapsto \log z + 2\pi i, \qquad \ell \longmapsto T^{-1}\ell = \exp(-2\pi i R)\ell,$$

and hence

$$\exp(R(\log z + 2\pi i))\exp(-2\pi i R)\ell = \exp(R \log z)\ell.$$

Definition 3.1 (Deligne's canonical extension)**.** The Deligne extension $\overline{\mathcal{L}}$ is the locally free $\mathcal{O}_\Delta$-module generated by the single-valued sections $\widetilde{\ell} = \exp(R\log z)\ell$, where ℓ runs over a basis of flat multivalued sections of $\mathbb{L}$.

The connection extends to $\overline{\mathcal{L}}$ with logarithmic pole at the origin. Indeed, since ℓ is flat, we compute

$$\begin{aligned} d(\exp(R\log z)\ell) &= d(\exp(R\log z))\ell + \exp(R\log z)d\ell \\ &= R\exp(R\log z)\ell\frac{dz}{z}. \end{aligned}$$

Thus

$$\nabla\widetilde{\ell} = R\widetilde{\ell}\frac{dz}{z},$$

so the residue of the logarithmic connection is precisely R.

3.3 Extension of the Hodge bundle

The subtle point is not merely extending the flat bundle, but extending the Hodge filtration subbundles. Let C be a compact Riemann surface, let $S \subset C$ be finite, and let $\mathbb{L}$ be a local system on $C \setminus S$. On $C \setminus S$, the associated flat bundle

$$(\mathcal{L}, \nabla) = (\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{C\setminus S}, \nabla)$$

comes with Hodge filtration subbundles $F^p\mathcal{L}$.

A priori, the extension of $F^p\mathcal{L}$ across S need not be visibly algebraic. It is not automatic from the definition of a holomorphic subbundle on $C \setminus S$ that its limit inside the Deligne extension is an algebraic subbundle. Schmid's nilpotent orbit theorem gives the required control: it shows that the Hodge filtration has a well-defined limiting filtration and hence extends across the boundary in a controlled algebraic manner. Simpson's Hermitian–Yang–Mills perspective gives a different conceptual explanation: in that framework, the nilpotent orbit theorem is reflected in precise metric estimates for harmonic bundles on non-compact curves.

We now recall the local form of Schmid's theorem. Choose a punctured disk $\Delta^* \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and lift the period map to the universal cover $\mathscr{H} \to \Delta^*$, where $q = \exp(2\pi i z)$ and $z \in \mathscr{H}$. We obtain a lifted period map

$$\widetilde{\tau} : \mathscr{H} \longrightarrow D,$$

where in the present example $D = \mathscr{H}$ and the compact dual is $\check{D} = \mathbb{P}^1$. If γ denotes the local monodromy, then

$$\widetilde{\tau}(z+1) = \gamma \cdot \widetilde{\tau}(z).$$

Since the family is semistable, γ is unipotent. Put

$$N := \log\gamma.$$

Define

$$\widetilde{\psi} : \mathscr{H} \longrightarrow \check{D}, \qquad \widetilde{\psi}(z) := \exp(-zN)\widetilde{\tau}(z).$$

Then $\widetilde{\psi}$ is invariant under $z \mapsto z+1$. Indeed,

$$\begin{aligned}\widetilde{\psi}(z+1) &= \exp(-(z+1)N)\widetilde{\tau}(z+1)\\ &= \exp(-(z+1)N)\gamma\widetilde{\tau}(z)\\ &= \exp(-(z+1)N)\exp(N)\widetilde{\tau}(z)\\ &= \exp(-zN)\widetilde{\tau}(z) = \widetilde{\psi}(z).\end{aligned}$$

Therefore $\widetilde{\psi}$ descends to a holomorphic map

$$\psi : \Delta^* \longrightarrow \check{D}.$$

Theorem 3.2 (Schmid's nilpotent orbit theorem, local form)**.** *The map $\psi : \Delta^* \to \check{D}$ extends holomorphically to*

$$\psi : \Delta \longrightarrow \check{D}.$$

Let $a := \psi(0)$. Then the nilpotent orbit

$$O(z) := \exp(zN)a$$

has the following properties. There exists $C > 0$ such that:

1. *if $\operatorname{Im} z \geq C$, then $O(z) \in D$;*
2. *for every $\varepsilon > 0$, there exists $A(\varepsilon) > 0$ such that*

$$d(O(z), \widetilde{\tau}(z)) \leq A(\varepsilon)\exp(-2\pi(1-\varepsilon)\operatorname{Im} z)$$

for all $\operatorname{Im} z \geq C$, where d is the invariant Hodge metric on D.

This theorem explains why the Hodge line extends algebraically. First by holomorphicity of ψ, we get a holomorphic extension of Hodge line bundle on Δ. Then by GAGA on the compact Riemann surface, every holomorphic line bundle is algebraic so that we get an algebraic extension[1].

3.4 Extension of the Higgs bundle

We now pass from the variation of Hodge structure to its associated logarithmic Higgs bundle. After taking Deligne's canonical extension and extending the Hodge filtration, we obtain

$$E^{1,0}_{\mathbb{P}^1} \subset \overline{\mathcal{V}}_{\mathbb{P}^1}, \qquad E^{0,1}_{\mathbb{P}^1} := \overline{\mathcal{V}}_{\mathbb{P}^1}/E^{1,0}_{\mathbb{P}^1}.$$

The logarithmic Gauss–Manin connection induces the Higgs field

$$\theta : E^{1,0}_{\mathbb{P}^1} \longrightarrow E^{0,1}_{\mathbb{P}^1} \otimes \Omega^1_{\mathbb{P}^1}(\log S).$$

Because the variation has weight one and comes from elliptic curves, the polarization identifies

$$E^{0,1} \simeq (E^{1,0})^\vee.$$

We will use the following two facts:

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[1] Remark. The original note explain the algebraicity as follows: as $q \to 0$ on Δ^*, equivalently $\operatorname{Im} z \to +\infty$ on the universal cover $\mathscr{H}$ (with $\exp(2\pi i z) = t$), the principal part of singularity of $\tau(t) = E^{1,0}_t$ is $\exp(\frac{1}{2\pi i}N\log t)$. By unipotent of N this is simply a polynomial in $\log t$, hence algebraic.

1. $\theta \neq 0$;
2. $\deg E^{1,0} > 0$.

The first fact follows from non-constancy of the period map. The second follows either from the Griffiths curvature formula or from the Yang–Mills–Higgs metric and polystability.

3.4.1 The Higgs field as the derivative of the period map

The Higgs field is the infinitesimal form of the period map. More precisely, Griffiths transversality gives

$$\nabla F^1 \subset F^0 \otimes \Omega^1_{\mathbb{P}^1}(\log S),$$

and the induced map on the quotient is

$$\theta : E^{1,0} \longrightarrow E^{0,1} \otimes \Omega^1_{\mathbb{P}^1}(\log S).$$

Equivalently, after contraction with tangent vectors, this gives

$$T_{\mathbb{P}^1}(-\log S) \longrightarrow \mathrm{Hom}(E^{1,0}, E^{0,1}),$$

which is precisely the differential $d\tau$ of the period map. Thus one may write, in this weight-one situation,

$$\theta = d\tau.$$

3.4.2 Why the two facts imply that $d\tau$ is an isomorphism

Since $|S| = 4$, we have

$$\Omega^1_{\mathbb{P}^1}(\log S) = \omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(S) \simeq \mathcal{O}_{\mathbb{P}^1}(-2+4) = \mathcal{O}_{\mathbb{P}^1}(2).$$

Write

$$E^{1,0}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(d).$$

Then

$$E^{0,1}_{\mathbb{P}^1} \simeq (E^{1,0}_{\mathbb{P}^1})^\vee \simeq \mathcal{O}_{\mathbb{P}^1}(-d).$$

The Higgs field is therefore a morphism

$$\theta : \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{\mathbb{P}^1}(2) = \mathcal{O}_{\mathbb{P}^1}(2-d),$$

or equivalently a section

$$\theta \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-2d)).$$

Since $\theta \neq 0$, we must have

$$2 - 2d \geq 0.$$

On the other hand, $\deg E^{1,0} = d > 0$, so $d \geq 1$. Hence $d = 1$.

Consequently,

$$E^{1,0} \simeq \mathcal{O}_{\mathbb{P}^1}(1), \qquad E^{0,1} \otimes \Omega^1_{\mathbb{P}^1}(\log S) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \simeq \mathcal{O}_{\mathbb{P}^1}(1).$$

Thus

$$\theta : \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$$

is a nonzero endomorphism of $\mathcal{O}_{\mathbb{P}^1}(1)$. Since

$$\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{C},$$

this morphism is multiplication by a nonzero scalar. Therefore θ is an isomorphism, and hence $d\tau$ is an isomorphism.

3.4.3 Proof of $\theta \neq 0$ and $\deg E^{1,0} > 0$

First, $\theta \neq 0$ because $\theta = d\tau$ and the period map τ is not constant.

It remains to show that $\deg E^{1,0} > 0$. Consider the logarithmic Higgs bundle

$$(E, \theta) := (E^{1,0} \oplus E^{0,1}, \theta).$$

Since the Higgs field has only the component

$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_{\mathbb{P}^1}(\log S)$$

and is zero on $E^{0,1}$, we have a Higgs subbundle

$$(E^{0,1}, 0) \hookrightarrow (E^{1,0} \oplus E^{0,1}, \theta).$$

The Hodge metric makes (E, θ) a polystable Higgs bundle with vanishing first Chern class; equivalently,

$$\deg(E) = \deg E^{1,0} + \deg E^{0,1} = 0.$$

By semistability of (E, θ), every Higgs subbundle has slope at most $\mu(E) = 0$. Applying this to $(E^{0,1}, 0)$ gives

$$\deg E^{0,1} \leq 0.$$

Since $\deg E^{1,0} + \deg E^{0,1} = 0$, it follows that

$$\deg E^{1,0} \geq 0.$$

We now exclude equality.

Suppose $\deg E^{0,1} = 0$. Then $(E^{0,1}, 0)$ has the same slope as the polystable Higgs bundle (E, θ). Hence it splits as a direct summand:

$$(E, \theta) \simeq (E^{0,1}, 0) \oplus (Q, \theta_Q),$$

where

$$Q = E/E^{0,1} \simeq E^{1,0}.$$

The induced Higgs field on the quotient is nilpotent. Since Q has rank one, any nilpotent endomorphism-valued one-form on Q must be zero; equivalently, a rank-one Higgs field which is nilpotent is identically zero. Therefore

$$\theta_Q = 0.$$

The above splitting then forces the original Higgs field to be zero, contradicting $\theta = d\tau \neq 0$. Hence

$$\deg E^{0,1} < 0,$$

and therefore

$$\deg E^{1,0} > 0.$$

Combining this with the previous subsection gives

$$E^{1,0} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$$

and shows that the derivative of the period map is an isomorphism. In the next lecture, we will recover this result from the viewpoint of moduli theory and identify

$$\rho\big(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})\big) = \Gamma(2).$$

Lecture 4: June 15, 2026

We now turn from the Legendre family to the general deformation-theoretic meaning of the derivative of a period map.

4.1 Kodaira–Spencer map, Hodge theory, and Higgs bundles

Given a variety, we have a complex structure. The complex structure is non-linear, and we hope to find a linear model and see the Hodge structure in the linear model, called the Hodge decomposition. We then study how it varies.

Let $f : X \to Y$ be a smooth family of (quasi)-compact complex manifolds. Pick a point whose fiber is denoted by the central fiber; we are going to study the variation of the complex structure near the central fiber.

The Kodaira–Spencer map has two constructions: the first one is algebraic and can be done over any field, and the second is a geometric construction over the complex numbers. We have the fundamental exact sequence

$$0 \to T_{X_0} \to T_X|_{X_0} \to f^*T_Y|_{\{0\}} \to 0.$$

For simplicity, we assume that the base is a curve. Then we see that $f^*T_Y|_{\{0\}}$ is a one-dimensional vector space spanned by $\mathbb{C}\frac{\partial}{\partial t}$. Giving a lifting map $s :\ f^*T_Y|_{\{0\}} \to T_X\Big|_{X_0}$ is the same as giving a holomorphic vector field $v \in H^0\left(X_0, T_X|_{X_0}\right)$ such that

$$s\left(\frac{\partial}{\partial t}\right) = v.$$

For this to be a splitting, we need

$$df(v) = \frac{\partial}{\partial t}.$$

Then every vector field $\xi \in T_X|_{X_0}$ decomposes uniquely as

$$\xi = (\xi - df(\xi)v) + df(\xi)v.$$

Here

$$df(\xi - df(\xi)v) = df(\xi) - df(\xi)df(v) = 0,$$

so

$$\xi - df(\xi)v \in T_{X_0}.$$

Thus $s :\ f^*T_Y|_{\{0\}} \to T_X\Big|_{X_0}$ induces a holomorphic splitting of the exact sequence.

The obstruction to lifting this vector field is measured by the extension class of the tangent sequence. By taking the long exact sequence

$$0 \to H^0(X_0, T_{X_0}) \to H^0(X_0, T_X|_{X_0}) \to H^0(X_0, f^*T_Y|_{\{0\}}) \xrightarrow{\delta} H^1(X_0, T_{X_0}) \to \cdots$$

induced by $(*)$, we know that

$$\tau = \delta(\partial_t) = 0 \iff \text{the exact sequence } (*) \text{ splits.}$$

We call τ the obstruction class, and $H^1(X_0, T_{X_0})$ is the group in which the extension class lies.

When we talk about the Higgs field or local Torelli, we will use this characterization, which is characteristic-free.

Next, we give a geometric characterization of the Kodaira–Spencer map. Geometrically, we can also construct such a class. Let

$$f : X_\Delta \to \Delta.$$

Then there is an open cover

$$X_\Delta = \bigcup \mathcal{U}_i$$

such that over any open neighborhood we have

$$\begin{tikzcd}
\mathcal{U}_i \arrow[rr, "\simeq"] \arrow[dr] & & U_i \times \Delta \arrow[dl] \\
& \Delta &
\end{tikzcd}$$

where $U_i = \mathcal{U}_i \cap X_0$, and $\simeq$ above is biholomorphic. Locally on $\mathcal{U}_i$, we have local holomorphic liftings, say

$$u_i \in H^0(\mathcal{U}_i, T_{X_\Delta}), \qquad u_j \in H^0(\mathcal{U}_j, T_{X_\Delta}).$$

The local liftings and their difference on an overlap are illustrated below. Restricting to the central

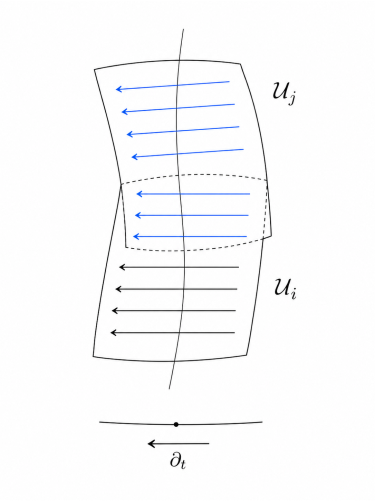

Figure 4.1: Local liftings used in the Čech construction of the Kodaira–Spencer class.

fiber gives

$$u_i|_{U_i} \in H^0\left(U_i, T_{X_\Delta}|_{X_0}\right).$$

By construction,

$$df\left(u_j - u_i\right) = df\left(u_j\right) - df\left(u_i\right) = 0.$$

Hence the difference is a vertical vector field. Therefore

$$\left(u_j - u_i\right)|_{U_i \cap U_j} \in H^0\left(U_i \cap U_j, T_{X_0}\right)$$

lies inside T_{X_0}, not merely in the sheaf of sections of $T_{X_\Delta}|_{X_0}$. Since $\left(u_j - u_i\right)|_{U_i \cap U_j} \in H^0\left(U_i \cap U_j, T_{X_0}\right)$ satisfies the cocycle condition,

$$\rho\left(\frac{\partial}{\partial t}\right) = \left[\{u_j - u_i\}\right] \in H^1\left(X_0, T_{X_0}\right)$$

gives the Kodaira–Spencer deformation class along $\frac{\partial}{\partial t}$.

Proposition 4.1. *The holomorphic vector field $\frac{\partial}{\partial t}$ lifts to a horizontal holomorphic vector field if and only if $\rho(\frac{\partial}{\partial t}) = 0$.*

Proof. Consider the following fundamental exact sequence

$$0 \to T_{X_0} \to T_X|_{X_0} \xrightarrow{df} f^*T_Y\Big|_{\{0\}} \to 0.$$

Then the Kodaira–Spencer class is just the extension class of this short exact sequence. Since the exact sequence splits if and only if the extension class is 0, this is equivalent to $\rho(\frac{\partial}{\partial t}) = 0$. ☐

We now consider the global Kodaira–Spencer map

$$\tau : T_Y \to R^1 f_* T_{X/Y},$$

which is the first connecting homomorphism of f_* applied to the fundamental exact sequence

$$0 \to T_{X/Y} \to T_X \to f^*T_Y \to 0.$$

We can rewrite the derivative θ of the period mapping of the Legendre family in terms of the Kodaira–Spencer map and the cup product.

$$\begin{tikzcd}
T_{\mathbb{P}^1}(\log S) \otimes R^0 f_* \Omega^1_{E/\mathbb{P}^1}(\log \Delta) \arrow[r, "\theta"] \arrow[d, "\tau\otimes id"'] & R^1 f_* \mathcal{O}_E \\
R^1 f_* T_{E/\mathbb{P}^1}(-\log S) \otimes R^0 f_* \Omega^1_{E/\mathbb{P}^1}(\log \Delta) \arrow[ur, "\cup"'] &
\end{tikzcd}$$

For the Legendre family, τ is an isomorphism, because $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the fine moduli space of elliptic curves with level-two structure. Therefore the tangent space at a point $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is naturally identified with

$$T_{Y,\lambda} \simeq H^1\left(E_\lambda, T_{E_\lambda}\right),$$

and hence the Kodaira–Spencer map is an isomorphism. Note that $\cup$ is an isomorphism because contraction with a nonzero holomorphic one-form identifies the relevant one-dimensional spaces.

Lecture 5: June 19, 2026

These notes discuss the Gauss–Manin connection and two constructions of graded Higgs bundles arising from geometry. The first is the classical system of Hodge bundles attached to a variation of Hodge structure. The second is the graded Higgs bundle arising from the Kodaira–Spencer map, in the spirit of Viehweg–Zuo. We also explain the Katz–Oda construction of the Gauss–Manin connection.

5.1 Overview

Let $Y = U \cup S$, where U is quasi-projective and S is SNC divisor, and let $f : X \to Y$ be a proper morphism which is smooth over $U = Y \setminus S$. Let Δ be the discriminant locus of f, for technical reasons, we also assume that $\Delta \to S$ is smooth.

We want to study two types of Higgs bundles attached to this family:

(1) The first construction is the system of Hodge bundles associated with the family, originating in Griffiths' theory and developed further in nonabelian Hodge theory. The Gauss–Manin connection is $\mathbb{C}$-linear and satisfies the Leibniz rule, but it is not $\mathcal{O}_Y$-linear. Passing to the associated graded of the Hodge filtration produces the $\mathcal{O}_Y$-linear Higgs field.

We may worry that if we take the grading, we may lose information. For example, suppose we have a filtered holomorphic vector bundle

$$0 \subset F^1 \subset F^0 = V.$$

Taking the associated graded gives

$$\mathrm{gr}_F V = F^1 \oplus V/F^1.$$

But we forget the extension class of the exact sequence

$$0 \to F^1 \to V \to V/F^1 \to 0.$$

With the additional harmonic metric, nonabelian Hodge theory relates this graded Higgs object back to the corresponding flat bundle.

This linear object is useful: the Higgs field is the derivative of the period map, and many hyperbolicity arguments can be formulated in terms of Higgs bundles. A limitation is that the chosen variation of Hodge structure can fail to detect some deformations of complex structure. This motivates the second construction.

(2) The second construction is a graded Higgs bundle extending the classical Kodaira–Spencer map. It measures variation of complex structure directly, although it is not defined purely from the underlying topological local system. It is particularly useful in applications to hyperbolicity.

The two constructions behave differently because dualization and higher direct image do not commute without additional hypotheses.

5.2 First construction: the system of Hodge bundles

Consider the logarithmic fundamental exact sequence

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0.$$

Taking exterior powers gives a natural filtration on $\Omega^p_X(\log \Delta)$. The relevant graded piece gives an exact sequence

$$0 \to f^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{X/Y}(\log \Delta) \to \operatorname{gr}^1 \Omega^p_X(\log \Delta) \to \Omega^p_{X/Y}(\log \Delta) \to 0.$$

Here

$$\operatorname{gr}^1 \Omega^p_X(\log \Delta) = \frac{\Omega^p_X(\log \Delta)}{f^*\Omega^2_Y(\log S) \wedge \Omega^{p-2}_X(\log \Delta)}.$$

Applying $R^q f_*$ and using the projection formula gives a connecting homomorphism

$$\theta^{p,q} \colon R^q f_* \Omega^p_{X/Y}(\log \Delta) \to \Omega^1_Y(\log S) \otimes R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log \Delta).$$

Define

$$E^{p,q} := R^q f_* \Omega^p_{X/Y}(\log \Delta).$$

For fixed weight k, the system of Hodge bundles is

$$\boxed{(E, \theta) = \left(\bigoplus_{p+q=k} E^{p,q}, \ \bigoplus_{p+q=k} \theta^{p,q} \right)}.$$

Thus

$$\theta^{p,q} \colon E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S).$$

The connecting homomorphism may also be expressed through the Kodaira–Spencer map. There is a diagram

$$\begin{tikzcd}
T_Y(-\log S) \otimes R^q f_* \Omega^p_{X/Y}(\log \Delta) \arrow[r, "\theta^{p,q}"] \arrow[d, "\tau\otimes\mathrm{id}"'] & R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log \Delta) \\
R^1 f_* T_{X/Y}(-\log \Delta) \otimes R^q f_* \Omega^p_{X/Y}(\log \Delta) \arrow[ur, "\cup"'] &
\end{tikzcd}$$

where

$$\tau \colon T_Y(-\log S) \to R^1 f_* T_{X/Y}(-\log \Delta)$$

is the logarithmic Kodaira–Spencer map. Hence $\theta^{p,q}$ is not an abstract morphism: it is obtained by composing the Kodaira–Spencer map with the cup product.

The Higgs field can be regarded as a morphism

$$\theta \colon T_Y(-\log S) \to \operatorname{End}(E).$$

It induces a Higgs field on $\mathrm{End}(E)$,

$$\operatorname{ad}\theta\colon\ \mathrm{End}(E)\to\mathrm{End}(E)\otimes\Omega^1_Y(\log S),\qquad \varphi\mapsto\theta\circ\varphi-\varphi\circ\theta.$$

More precisely, the formula is interpreted using the wedge product with logarithmic forms. Since the Higgs field satisfies

$$\theta\wedge\theta=0,$$

we have $[\theta_v,\theta_w]=0$ for local logarithmic vector fields $v,w\in T_Y(-\log S)$. Therefore

$$\theta\big(T_Y(-\log S)\big)\subseteq\ker(\operatorname{ad}\theta).$$

This inclusion is important in hyperbolicity arguments. Under the Hodge metric, kernels of Higgs fields carry seminegativity properties; one can then transfer this negativity to subsheaves related to $T_Y(-\log S)$.

Using the inclusion

$$\mathrm{Im}(\theta)\subseteq\ker(\operatorname{ad}\theta),$$

one obtains a morphism of complexes

$$T_Y(-\log S)[0]\to\Big[\mathrm{End}(E)\xrightarrow{\operatorname{ad}\theta}\mathrm{End}(E)\otimes\Omega^1_Y(\log S)\xrightarrow{\operatorname{ad}\theta}\mathrm{End}(E)\otimes\Omega^2_Y(\log S)\to\cdots\Big].$$

Passing to hypercohomology gives a map

$$H^1\big(Y,T_Y(-\log S)\big)\to\mathbb{H}^1_{\mathrm{Higgs}}\big(Y,\mathrm{End}(E)\big).$$

The target is naturally interpreted as the tangent space to a Dolbeault moduli space at the Higgs bundle (E,θ). If Y itself varies in a family over a base B, then the ordinary Kodaira–Spencer map

$$T_0B\to H^1\big(Y,T_Y(-\log S)\big)$$

can be composed with the above map. The resulting map

$$T_0B\to\mathbb{H}^1_{\mathrm{Higgs}}\big(Y,\mathrm{End}(E)\big)$$

is often called the non-abelian Kodaira–Spencer map, which can be used to study Esnault–Kerr conjecture.

5.3 Second construction: the graded Higgs bundle from the Kodaira–Spencer map

The local Torelli problem asks whether the infinitesimal period map is injective. In many situations the infinitesimal period map has a kernel, or even vanishes, so the classical Higgs field does not see all deformations of the complex structure. This motivates the second construction.

The second construction is the graded Higgs bundle arising directly from the Kodaira–Spencer map. It does not refer directly to a variation of Hodge structure. Its starting point is the logarithmic tangent exact sequence

$$0\to T_{X/Y}(-\log\Delta)\to T_X(-\log\Delta)\to f^*T_Y(-\log S)\to 0.$$

Taking exterior powers gives an exact sequence

$$0 \to \bigwedge^{p} T_{X/Y}(-\log \Delta) \to \operatorname{gr}^0 \bigwedge^{p} T_X(-\log \Delta) \to \bigwedge^{p-1} T_{X/Y}(-\log \Delta) \otimes f^*T_Y(-\log S) \to 0.$$

For brevity write

$$T^p_{X/Y}(-\log \Delta) := \bigwedge^{p} T_{X/Y}(-\log \Delta).$$

The connecting homomorphism gives maps

$$\tau^p \colon R^{p-1} f_* T^{p-1}_{X/Y}(-\log \Delta) \otimes T_Y(-\log S) \to R^p f_* T^p_{X/Y}(-\log \Delta).$$

Equivalently, if

$$G^p := R^p f_* T^p_{X/Y}(-\log \Delta),$$

then the maps τ^p define a Higgs field

$$\tau^p \colon G^{p-1} \to G^p \otimes \Omega^1_Y(\log S).$$

For $p = 1$ this is exactly the logarithmic Kodaira–Spencer map

$$\tau^1 \colon T_Y(-\log S) \to R^1 f_* T_{X/Y}(-\log \Delta).$$

If the family is infinitesimally non-isotrivial at a point, then the Kodaira–Spencer map is nonzero there.

This construction is algebraic. To make it useful in geometry, one needs positivity or negativity results, usually obtained by Hodge-theoretic or analytic methods.

Proposition 5.1 (Negativity properties)**.** *Assume the family satisfies the standard hypotheses under which the Hodge metric and the Viehweg–Zuo construction apply. Then:*

(1) $\ker \theta$ *is seminegative in the appropriate curvature or degree sense;*

(2) $\ker \tau$ *satisfies an analogous seminegativity statement, with strict negativity under suitable maximal-variation hypotheses.*

Remark 5.2. These negativity statements do not follow from formal homological algebra alone. They require some transcendental approach.

5.4 The system of Hodge bundles as the associated graded of the filtered de Rham bundle

We now explain how the system of Hodge bundles is obtained as the associated graded object of the filtered de Rham bundle. The key input is E_1-degeneration of the Hodge-to-de Rham spectral sequence.

For simplicity, we omit the log structure. The fundamental exact sequence gives

$$0 \to f^*\Omega^1_Y \otimes \Omega^{p-1}_{X/Y} \to \operatorname{gr}^1 \Omega^p_X \to \Omega^p_{X/Y} \to 0.$$

In terms of complexes, this becomes

$$0 \to f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1] \to \operatorname{gr}^1 \Omega^\bullet_X \to \Omega^\bullet_{X/Y} \to 0.$$

Taking derived direct images gives a long exact sequence

$$\cdots \to \mathbb{R}^k f_* \operatorname{gr}^1 \Omega^\bullet_X \to \mathbb{R}^k f_* \Omega^\bullet_{X/Y} \xrightarrow{\partial} \mathbb{R}^{k+1} f_*(f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1]) \to \cdots .$$

By the projection formula,

$$\mathbb{R}^{k+1} f_*(f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1]) \simeq \Omega^1_Y \otimes \mathbb{R}^k f_* \Omega^\bullet_{X/Y}.$$

Thus the connecting homomorphism is the Gauss–Manin connection

$$\nabla^{\mathrm{GM}} \colon \mathcal{V}^k \to \mathcal{V}^k \otimes \Omega^1_Y, \qquad \mathcal{V}^k := \mathbb{R}^k f_* \Omega^\bullet_{X/Y}.$$

In the logarithmic setting the same construction gives

$$\nabla^{\mathrm{GM}} \colon \mathcal{V}^k \to \mathcal{V}^k \otimes \Omega^1_Y(\log S).$$

Consider the truncation complex

$$\sigma^{\geq p}\Omega^\bullet_{X/Y} := [\cdots \to 0 \to \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y} \to \cdots].$$

Define the Hodge filtration bundle by

$$F^p\mathcal{V}^k := \operatorname{Im}\left[\mathbb{R}^k f_* \sigma^{\geq p}\Omega^\bullet_{X/Y} \to \mathbb{R}^k f_* \Omega^\bullet_{X/Y}\right].$$

This gives a decreasing filtration

$$\mathcal{V}^k = F^0\mathcal{V}^k \supset F^1\mathcal{V}^k \supset \cdots \supset F^{k+1}\mathcal{V}^k = 0.$$

Griffiths transversality says

$$\nabla^{\mathrm{GM}}(F^p\mathcal{V}^k) \subset F^{p-1}\mathcal{V}^k \otimes \Omega^1_Y(\log S).$$

Therefore the associated graded object carries a Higgs field

$$\theta \colon \operatorname{gr}^p_F \mathcal{V}^k \to \operatorname{gr}^{p-1}_F \mathcal{V}^k \otimes \Omega^1_Y(\log S),$$

where

$$\operatorname{gr}^p_F \mathcal{V}^k := F^p\mathcal{V}^k / F^{p+1}\mathcal{V}^k.$$

Thus we constructed a graded Higgs bundle

$$\boxed{\operatorname{gr}_{F^\bullet}(\mathcal{V}^k, \nabla^{\mathrm{GM}}) = \left(\bigoplus_{p=0}^{k} F^p\mathcal{V}^k/F^{p+1}\mathcal{V}^k, \bigoplus_{p=0}^{k} \theta_p\right).}$$

The crucial point is to identify the graded pieces $F^p\mathcal{V}^k/F^{p+1}\mathcal{V}^k$ with the Hodge bundles $R^{k-p}f_*\Omega^p_{X/Y}$. This is where E_1-degeneration enters.

Proposition 5.3. *Let*

$$E_1^{p,q} = R^q f_* \Omega^p_{X/Y} \Longrightarrow \mathbb{R}^{p+q} f_* \Omega^\bullet_{X/Y} = \mathcal{V}^{p+q}$$

be the Hodge-to-de Rham spectral sequence. Then:

(1) In general,

$$\operatorname{rk} \mathcal{V}^k \le \sum_{p+q=k} \operatorname{rk} R^q f_* \Omega^p_{X/Y}.$$

(2) If equality holds, equivalently if the spectral sequence degenerates at E_1 in total degree k, then

$$F^p \mathcal{V}^k / F^{p+1} \mathcal{V}^k \simeq R^{k-p} f_* \Omega^p_{X/Y}.$$

Moreover, the natural map

$$\mathbb{R}^k f_* \sigma^{\ge p} \Omega^\bullet_{X/Y} \to F^p \mathcal{V}^k$$

is an isomorphism, and hence

$$\mathbb{R}^k f_* \sigma^{\ge p} \Omega^\bullet_{X/Y} \to \mathbb{R}^k f_* \Omega^\bullet_{X/Y}$$

is injective with image $F^p \mathcal{V}^k$.

Proof. For a spectral sequence of locally free sheaves, the abutment carries a filtration whose graded pieces are $E_\infty^{p,q}$. Hence

$$\operatorname{rk} \mathcal{V}^k = \sum_{p+q=k} \operatorname{rk} E_\infty^{p,q} \le \sum_{p+q=k} \operatorname{rk} E_1^{p,q} = \sum_{p+q=k} \operatorname{rk} R^q f_* \Omega^p_{X/Y}.$$

This proves (1).

Assume equality holds. Since each $E_r^{p,q}$ is obtained from $E_1^{p,q}$ by taking successive kernels and quotients, we have

$$\operatorname{rk} E_\infty^{p,q} \le \operatorname{rk} E_1^{p,q}.$$

The equality of the sums in total degree k therefore forces

$$\operatorname{rk} E_\infty^{p,q} = \operatorname{rk} E_1^{p,q} \qquad \text{for all } p+q=k.$$

Thus no rank is lost along the spectral sequence in total degree k. Equivalently, the spectral sequence degenerates at E_1 in this total degree. Therefore

$$E_\infty^{p,k-p} = \operatorname{gr}^p_F \mathcal{V}^k \simeq E_1^{p,k-p} = R^{k-p} f_* \Omega^p_{X/Y}.$$

It remains to justify the injectivity statement for the truncated complex. The truncated complex has a spectral sequence

$${}^{(p)}E_1^{a,b} = \begin{cases} R^b f_* \Omega^a_{X/Y}, & a \ge p, \\ 0, & a < p, \end{cases} \Longrightarrow \mathbb{R}^{a+b} f_* \sigma^{\ge p} \Omega^\bullet_{X/Y}.$$

Since the original spectral sequence degenerates at E_1 in total degree k, the truncated one also degenerates in total degree k. Hence

$$\operatorname{rk} \mathbb{R}^k f_* \sigma^{\ge p} \Omega^\bullet_{X/Y} = \sum_{\substack{a+b=k \\ a \ge p}} \operatorname{rk} R^b f_* \Omega^a_{X/Y}.$$

On the other hand,

$$\operatorname{rk} F^p\mathcal{V}^k = \sum_{a\geq p} \operatorname{rk} \operatorname{gr}_F^a \mathcal{V}^k = \sum_{\substack{a+b=k\\ a\geq p}} \operatorname{rk} R^b f_*\Omega^a_{X/Y}.$$

By definition the map

$$\mathbb{R}^k f_*\sigma^{\geq p}\Omega^\bullet_{X/Y} \to F^p\mathcal{V}^k$$

is surjective. Since the two sides have the same rank and are locally free under the present hypotheses, it is an isomorphism. This proves the desired injectivity into $\mathcal{V}^k$. □

5.5 A complex analytic proof of E_1-degeneration

We now recall the standard complex analytic proof of E_1-degeneration for a smooth proper Kähler morphism. For simplicity, assume that $f\colon X \to Y$ is smooth and proper, with compact Kähler fibers.

Proposition 5.4 (E_1-degeneration)**.** *For a smooth proper Kähler morphism $f\colon X \to Y$, the Hodge-to-de Rham spectral sequence*

$$E_1^{p,q} = R^q f_*\Omega^p_{X/Y} \Longrightarrow \mathbb{R}^{p+q} f_*\Omega^\bullet_{X/Y}$$

degenerates at E_1.

Proof. For each $s \in Y$, Hodge theory on the compact Kähler manifold X_s gives the Hodge decomposition

$$H^n_{\mathrm{dR}}(X_s,\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X_s,\Omega^p_{X_s}).$$

Therefore

$$b_n(X_s) = \sum_{p+q=n} h^{p,q}(X_s).$$

The Betti numbers $b_n(X_s)$ are locally constant in a smooth proper family, because the fibers are locally topologically trivial by Ehresmann's theorem. The functions

$$s \mapsto h^{p,q}(X_s)$$

are upper semicontinuous. Since their sum is locally constant and, for compact Kähler manifolds, Hodge symmetry and Hodge decomposition identify these dimensions with the ranks of the Hodge pieces, the Hodge numbers are locally constant in a smooth proper Kähler family. By Grauert's theorem, the sheaves

$$R^q f_*\Omega^p_{X/Y}$$

are locally free and their formation commutes with base change. Moreover,

$$\left(R^q f_*\Omega^p_{X/Y}\right) \otimes \kappa(s) \simeq H^q(X_s,\Omega^p_{X_s}).$$

Similarly,

$$\mathcal{V}^n := \mathbb{R}^n f_*\Omega^\bullet_{X/Y}$$

is the holomorphic vector bundle associated to the flat local system $R^n f_*\mathbb{C}$, and

$$\operatorname{rk} \mathcal{V}^n = b_n(X_s).$$

Thus

$$\operatorname{rk} \mathcal{V}^n = b_n(X_s) = \sum_{p+q=n} h^{p,q}(X_s) = \sum_{p+q=n} \operatorname{rk} R^q f_* \Omega^p_{X/Y}.$$

By the rank criterion in the previous proposition, the Hodge-to-de Rham spectral sequence degenerates at E_1. □

Lecture 6: June 22, 2026

Today, we will talk about the classifying space of polarized Hodge structures.

6.1 Polarized variations of Hodge structure and systems of Hodge bundles

Consider the free lattice

$$\mathbb{V}_{\mathbb{Z}} \subset \mathbb{V}_{\mathbb{R}} \subset \mathbb{V}_{\mathbb{C}}.$$

We can define a Hodge structure by

$$\mathbb{V}_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}.$$

Equivalently, it is given by a decreasing filtration

$$V_{\mathbb{C}} = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0$$

with

$$F^p = \bigoplus_{i \geq p} H^{i,k-i}, \qquad V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}} \quad \text{for all } p.$$

In this notation,

$$H^{p,q} = F^p \cap \overline{F^q}, \qquad p+q=k.$$

We can also define a polarization on the Hodge structure. A polarization is a non-degenerate bilinear form

$$S : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \longrightarrow \mathbb{Z}$$

which is symmetric if k is even and skew-symmetric if k is odd. Most importantly, it satisfies the Hodge–Riemann bilinear relations:

1. $S(H^{p,q}, H^{r,s}) = 0$ unless $(p,q) = (s,r)$;
2. for every non-zero $v \in H^{p,q}$,

$$i^{p-q} S(v, \overline{v}) > 0.$$

Note that we can also define Hodge structures and polarized Hodge structures using the Deligne torus. This will be useful when studying arithmetic properties.

We can then study variations of polarized Hodge structure (PVHS). Let M be a connected complex manifold.

A variation of polarized Hodge structure (PVHS) of weight k on M consists of the following data:

1. A holomorphic vector bundle $\mathcal{V}$ on M equipped with a flat holomorphic connection, called the Gauss–Manin connection,

   $$\nabla : \mathcal{V} \longrightarrow \Omega^1_M \otimes \mathcal{V}, \qquad \nabla^2 = 0.$$

   This corresponds to a complex local system

   $$\mathbb{V}_{\mathbb{C}} = \ker \left\{ \nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_M \right\},$$

   which carries a $\mathbb{Z}$-structure, or more generally a $\mathbb{Q}$-structure.

2. A polarization, namely a flat non-degenerate bilinear form

   $$S : \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{O}_M.$$

3. A Hodge filtration on the holomorphic bundle, namely a decreasing filtration by holomorphic subbundles

   $$\mathcal{V} = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0.$$

For every $t \in M$, the fiber $\left(\mathcal{V}_t, F^\bullet_t, S_t\right)$ is a polarized Hodge structure. Moreover, the filtration satisfies Griffiths transversality:

$$\nabla F^p \subset \Omega^1_M \otimes F^{p-1}, \quad \forall p.$$

Remark 6.1. In the definition above, we use a holomorphic bundle with a flat holomorphic connection. Giving a flat holomorphic connection on a holomorphic vector bundle is equivalent to giving a flat smooth connection on the underlying complex bundle, as in the following one-to-one correspondence:

$$(\mathcal{E}, D = \nabla^{1,0}) \Longleftrightarrow (E, \nabla = D + \bar{\partial}_E = \nabla^{1,0} + \nabla^{0,1}).$$

Here the left-hand side is a holomorphic bundle with a flat holomorphic connection, while the right-hand side is a smooth complex vector bundle with a flat smooth connection.

Note that we can define

$$\mathcal{H}^{p,k-p} = F^p \cap \overline{F^{k-p}},$$

which is a smooth, or rather real analytic, subbundle. It is not holomorphic because we take conjugation. This is a genuine difference between real analytic and holomorphic objects.

We then linearize the problem.

Let

$$E^{p,q} := F^p / F^{p+1}, \qquad p + q = k,$$

and put

$$E := \bigoplus_{p+q=k} E^{p,q} = \mathrm{gr}_F(\mathcal{V}).$$

By Griffiths transversality, the connection induces maps

$$\theta^{p,q} : E^{p,q} \longrightarrow \Omega^1_M \otimes E^{p-1,q+1}.$$

Their direct sum

$$\theta = \bigoplus_{p+q=k} \theta^{p,q} : E \longrightarrow \Omega^1_M \otimes E$$

is the Higgs field of the PVHS. Thus

$$(E,\theta)=\mathrm{gr}_F(\mathcal{V},\nabla)$$

gives the system of graded Higgs bundles. The flatness condition $\nabla^2=0$ implies the integrability condition $\theta\wedge\theta=0$. Note that all the graded pieces $E^{p,q}$ are holomorphic vector bundles, which are smoothly isomorphic to the real analytic bundles

$$E^{p,q}\cong\mathcal{H}^{p,q}.$$

6.2 Classifying space of polarized Hodge structures

Fix a free lattice, which induces real and complex vector spaces

$$\mathbb{V}_{\mathbb{Z}}\subset\mathbb{V}_{\mathbb{R}}\subset\mathbb{V}_{\mathbb{C}},$$

and fix Hodge numbers $h^{p,q}$, $p+q=k$, such that

$$\dim\mathbb{V}_{\mathbb{C}}=\sum_{p+q=k}h^{p,q}.$$

Consider the flag variety

$$\check{\mathcal{F}}=\left\{V_{\mathbb{C}}=F^0\supset F^1\supset\cdots\supset F^k\supset\{0\}\;\middle|\;\dim F^p=\sum_{i\geq p}h^{i,k-i}\right\}.$$

It parametrizes flags of linear subspaces with prescribed dimensions. One can check that it is a closed subvariety of the product of Grassmannians

$$\prod_{i=0}^{k}\mathrm{Grass}\left(\mathbb{V}_{\mathbb{C}},\dim F^i\right).$$

It is easy to check that the $\mathrm{GL}(V_{\mathbb{C}})$-action on the Grassmannian restricts to $\check{\mathcal{F}}$.

We then add extra conditions:

1. $V_{\mathbb{C}}=F^p\oplus\overline{F}^{k-p+1}$, which is a real algebraic condition and also an open condition, since it can be converted into the non-vanishing of a determinant. We denote by $\mathcal{F}\subset\check{\mathcal{F}}$ the set of all filtrations satisfying this condition.
2. The Hodge–Riemann bilinear relation

$$S\left(F^p,F^{k-p+1}\right)=0,$$

   which is an algebraic condition. We denote by $\check{D}\subset\check{\mathcal{F}}$ the set of all filtrations satisfying this condition.
3. The polarization condition

$$S(Cv,\overline{v})>0\qquad\text{for any } v\neq 0,$$

   where C is the Weil operator satisfying $Cv=i^{p-q}v$ for $v\in H^{p,q}$. This is a semi-algebraic condition. We denote by $D\subset\check{D}$ the set of filtrations satisfying these conditions.

Hence we have the following inclusion relation:

$$\begin{tikzcd}
D \arrow[r] \arrow[d, hook] & \mathcal{F} \arrow[d, hook] & \\
\check{D} \arrow[r, hook] & \check{\mathcal{F}} \arrow[r, hook] & \prod_{j=0}^{k} \operatorname{Grass}\left(\mathbb{V}_{\mathbb{C}}, \sum_{i \geq j} h^{i,k-i}\right)
\end{tikzcd}$$

The horizontal arrows are closed embeddings, while the vertical inclusions are open embeddings. Thus D is locally closed in the product of Grassmannians.

We call D the period domain, or the classifying space of polarized Hodge structures. It turns out that D is a complex manifold which parametrizes Hodge structures on $\mathbb{V}_{\mathbb{C}}$ of weight k, with Hodge numbers $h^{p,q}$, polarized by S. Thus it is a moduli space of Hodge structures.

Remark 6.2. The Deligne torus gives a finer description. It can be used to define smaller moduli spaces, called Mumford–Tate subvarieties. The period domain is the largest Mumford–Tate subvariety.

There is a natural group action on the compact dual $\check{D}$. Define

$$G_{\mathbb{C}} = \{ g \in \operatorname{GL}(\mathbb{V}_{\mathbb{C}}) \mid S(gu, gv) = S(u,v) \}.$$

It is easy to check that $G_{\mathbb{C}}$ preserves $\check{D}$. The non-trivial fact, proved by Griffiths, is that it acts transitively on $\check{D}$. Similarly,

$$G_{\mathbb{R}} = \{ g \in \operatorname{GL}(V_{\mathbb{R}}) \mid S(gu, gv) = S(u,v) \}$$

preserves D and acts transitively on D.

Let $o \in D$ be a reference point with reference Hodge filtration F_o^p, and let

$$B = \{ g \in G_{\mathbb{C}} \mid gF_o^p \subset F_o^p \text{ for all } p \}.$$

Then

$$D \cong G_{\mathbb{R}}/V, \qquad \check{D} \cong G_{\mathbb{C}}/B,$$

where $V = G_{\mathbb{R}} \cap B$. From this, we see that D and $\check{D}$ are non-singular. Note that D carries a $G_{\mathbb{R}}$-invariant metric called the Hodge metric.

The subgroup V is not necessarily maximal. If it is maximal, then D is a symmetric space. In general, D lies over a symmetric space.

Remark 6.3. We can also define

$$G_{\mathbb{Z}} = \{ g \in \operatorname{GL}(V_{\mathbb{Z}}) \mid S(gu, gv) = S(u,v) \}.$$

When the PVHS comes from geometry, for example from the Legendre family, the monodromy group lies in $G_{\mathbb{Z}}$.

6.3 Period mappings

Let $(\mathcal{V}, \nabla, F^\bullet, S)$ be a PVHS on a connected complex manifold M. Let

$$\pi : \widetilde{M} \longrightarrow M$$

be the universal cover. Since $\pi^*\mathcal{V}$ is flat and $\widetilde{M}$ is simply connected, the local system becomes trivial:

$$\pi^*\mathcal{V} \simeq V_{\mathbb{C}} \times \widetilde{M}.$$

Under this trivialization, the Gauss–Manin connection becomes the ordinary differential, and the pulled-back Hodge filtration becomes a holomorphic family of filtrations

$$\pi^* F^\bullet \subset V_{\mathbb{C}} \times \widetilde{M}.$$

Since these filtrations are polarized by S, one obtains a holomorphic map

$$\widetilde{\Phi} : \widetilde{M} \longrightarrow D,$$

called the lifted period map. It is equivariant with respect to the monodromy representation

$$\rho : \pi_1(M) \longrightarrow G_{\mathbb{Z}},$$

so it descends to a map

$$\Phi : M \longrightarrow \Gamma \backslash D, \qquad \Gamma := \rho(\pi_1(M)).$$

6.4 Derivative of the period map

We want to study the derivative of the period map, and hence we need to study the tangent space of the period domain. There are two approaches to describing the tangent space. The first uses the observation that D is a homogeneous space, so we can use Lie algebras. The second uses the observation that D is a locally closed subvariety of a Grassmannian, whose tangent space has a very good description. From this approach, we can immediately see that the derivative of the period map is given by the Higgs field.

Consider the inclusion

$$\iota : \check{D} \hookrightarrow \prod_{i=1}^{k} \mathrm{Grass}\left(\mathbb{V}_{\mathbb{C}}, f_i\right),$$

where $f_i = \dim F_o^i$. Taking the derivative, we get

$$\mu = d\iota : T_o\check{D} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{b} \hookrightarrow \bigoplus_{p=1}^{k} \mathrm{Hom}\left(F_o^p, \mathbb{V}_{\mathbb{C}}/F_o^p\right).$$

We next study the relation between the derivative of the period map, the Gauss–Manin connection, and the Higgs field.

6.4.1 Relation with the Gauss–Manin connection

Proposition 6.4. *Let ξ be a germ of a holomorphic vector field at $x \in \widetilde{M}$. For each p, define*

$$z_p(\xi) : F^p_x \longrightarrow V_{\mathbb{C}}/F^p_x$$

by

$$z_p(\xi)(s) := \nabla_\xi(\widetilde{s}) \bmod F^p_x,$$

where $\widetilde{s}$ is any local holomorphic section of F^p extending s, and ∇ is the Gauss–Manin connection. Then one can check that the derivative of the period map

$$d\widetilde{\Phi}_x : T_x\widetilde{M} \longrightarrow \widetilde{\Phi}^* T_D$$

can be described by the Gauss–Manin connection as

$$\mu(d\widetilde{\Phi}_x(\xi)) = (z_1(\xi), z_2(\xi), \ldots, z_k(\xi)) .$$

Proof. Omitted. ☐

We go one step further to the Higgs field. We have the following inclusion:

$$\bigoplus_{p=1}^{k} \mathrm{Hom}\left(F^p/F^{p+1}, F^{p-1}/F^p\right) \subset \bigoplus_{p=1}^{k} \mathrm{Hom}\left(F^p, V_{\mathbb{C}}/F^p\right).$$

Indeed, given

$$u \in \mathrm{Hom}\left(F^p/F^{p+1}, F^{p-1}/F^p\right),$$

define a map

$$\widetilde{u} : F^p \longrightarrow V_{\mathbb{C}}/F^p$$

by the composition

$$F^p \longrightarrow F^p/F^{p+1} \xrightarrow{u} F^{p-1}/F^p \hookrightarrow V_{\mathbb{C}}/F^p.$$

By Griffiths transversality,

$$\nabla_\xi F^p \subset F^{p-1},$$

so the Gauss–Manin connection ∇^p_ξ descends to the maps

$$\overline{\nabla}^p_\xi : F^p/F^{p+1} \longrightarrow F^{p-1}/F^p.$$

These are precisely the components of the Higgs field:

$$\theta^{p,k-p}(\xi) : E^{p,k-p} \longrightarrow E^{p-1,k-p+1}.$$

In summary, we have the following picture for the derivative of the period map. The derivative of the period map is precisely the Higgs field.

$$\begin{tikzcd}
T\widetilde{M} \arrow[r, "d\widetilde{\Phi}"] \arrow[dr, "\theta"'] & \bigoplus_{p=1}^{k} \operatorname{Hom}\left(F^p, V_{\mathbb{C}}/F^p\right) \\
 & \bigoplus_{p=1}^{k} \operatorname{Hom}\left(F^p/F^{p+1}, F^{p-1}/F^p\right) \arrow[u, hook]
\end{tikzcd}$$

We want

$$\theta : T\widetilde{M} \longrightarrow \bigoplus_{p=1}^{k} \operatorname{Hom}\left(F^p/F^{p+1}, F^{p-1}/F^p\right)$$

to be generically injective. This is the local Torelli problem.

6.5 Local Torelli

Let $(\mathcal{V}, \nabla, F^\bullet, S)$ be a PVHS on M, and let (E, θ) be the graded Higgs bundle. Let

$$\Phi : M \longrightarrow \Gamma \backslash D$$

be the period map induced by the PVHS.

Definition 6.5. The variation satisfies local Torelli at a point $x \in M$ if the differential

$$d\Phi_x : T_{M,x} \longrightarrow T_{\Phi(x)}(\Gamma \backslash D)$$

is injective. It satisfies generic local Torelli if $d\Phi$ is injective at a general point of M.

Since $d\Phi$ is represented by the Higgs field, as discussed in the previous section, local Torelli is equivalent to the injectivity of

$$T_M \longrightarrow \bigoplus_{p} \operatorname{Hom}\left(E^{p,k-p}, E^{p-1,k-p+1}\right).$$

Typically, the map

$$T_M \longrightarrow \left(E^{k,0}\right)^{\vee} \otimes E^{k-1,1}$$

is injective. Let us consider the geometric situation of a smooth family of projective varieties, and consider the middle-dimensional variation of Hodge structure. Let $f : X \to M$ be the universal family of smooth projective varieties, and consider the system of Hodge bundles

$$(E, \theta) = \left(\bigoplus_{p+q=k} R^q f_* \Omega^p_{X/M}, \bigoplus_{p+q=k} \theta^{p,q} \right).$$

Then consider the following diagram, which writes the Higgs field as the composition of the Kodaira–Spencer map with the cup product mentioned last time:

$$\begin{tikzcd}
T_M \otimes R^0 f_* \Omega^k_{X/M} \arrow[r, "\theta^{k,0}"] \arrow[d, hook] & R^1 f_* \Omega^{k-1}_{X/M} \\
R^1 f_* T_{X/M} \otimes R^0 f_* \Omega^k_{X/M} \arrow[ur, "\cup"'] &
\end{tikzcd}$$

Here the Kodaira–Spencer map is injective for a universal family of projective manifolds. On the other hand, if we have

$$\Omega^k_{X/M}\big|_{X_t} \simeq \mathcal{O}_{X_t},$$

then the cup product is injective. This happens for families of elliptic curves and families of Calabi–Yau manifolds.

For the surjectivity of the cup product, this is related to Bogomolov–Tian–Todorov.

For hypersurfaces of high degree, we can also prove injectivity. This is the work of Peters and Steenbrinks.