Hodge Theory Course Notes Summer 2026

Lecture 1 — 2026-06-05

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One of the main goals of this course is to use Hodge theory to understand the global structure of moduli spaces.

We begin with a very simple example. Today's lecture has two main topics: (1) the geometry of $\mathbb{P}^1 \setminus \{0,1,\infty\}$, the simplest hyperbolic Riemann surface, and (2) the Hodge structure on elliptic curves.

1 Geometry of $\mathbb{P}^1 \setminus \{0,1,\infty\}$

Let us consider the parameter space

$$\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}.$$

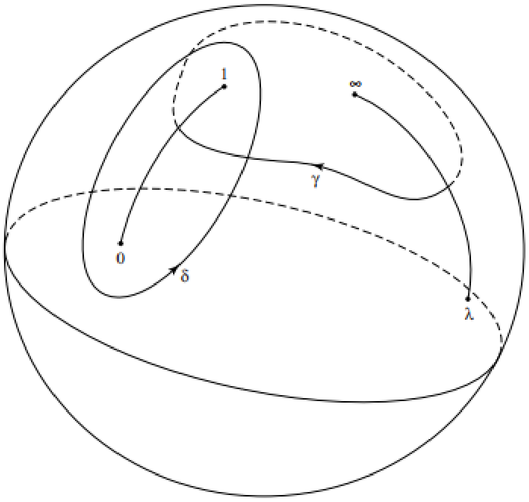
Our story starts from the following Picard–Fuchs equation:

$$\lambda(\lambda-1)\pi'' + (2\lambda-1)\pi' + \frac{1}{4}\pi = 0.$$

This ordinary differential equation has regular singularities at $\{0,1,\infty\}$, and outside these points its solution space is a two-dimensional vector space. A local basis of solutions is given by

$$A(\lambda) = \int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \qquad B(\lambda) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}},$$

where γ and δ are the loops on the Riemann sphere shown below.



These functions are called the *periods* of the Legendre family. On any disk

$$\Delta \subset \mathbb{P}^1 \setminus \{0, 1, \infty\},$$

the periods are single-valued holomorphic functions of λ. However, if we analytically continue them along a loop α around one of the points in $\{0, 1, \infty\}$ and then return to the original value of λ, the resulting solutions may differ from the original ones:

$$(A', B') = T_\alpha(A, B),$$

where T_α is called the monodromy transformation. It turns out that

$$T_\alpha \in SL_2(\mathbb{Z}).$$

We then define the period map by

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \longrightarrow \mathbb{C}, \qquad \lambda \longmapsto \frac{B(\lambda)}{A(\lambda)}.$$

One checks that the image of this period map lies in the upper half-plane

$$\mathscr{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

and that the map is holomorphic. Passing to the universal cover, we obtain a holomorphic map

$$\widetilde{\tau} : \mathbb{P}^1 \setminus \widetilde{\{0, 1, \infty\}} \longrightarrow \mathscr{H}.$$

One can verify, for example by following the original approach of Abel and Jacobi and computing the derivative of $\widetilde{\tau}$, that $\widetilde{\tau}$ is a biholomorphism onto $\mathscr{H}$.

It is worth mentioning that Deligne and Mostow considered an analogous problem in much greater generality, studying complements of hyperplane arrangements in projective spaces.

The biholomorphism $\widetilde{\tau}$ is equivariant: the domain carries the natural action of

$$\pi_1\big(\mathbb{P}^1 \setminus \{0, 1, \infty\}\big),$$

while on $\mathscr{H}$ we have the action induced by the monodromy representation

$$\rho : \pi_1\big(\mathbb{P}^1 \setminus \{0, 1, \infty\}\big) \longrightarrow \operatorname{Aut}(V_\lambda), \qquad \alpha \longmapsto T_\alpha,$$

where V_λ is the two-dimensional solution space. The image satisfies

$$\rho\big(\pi_1\big(\mathbb{P}^1 \setminus \{0, 1, \infty\}\big)\big) = \Gamma(2) = \ker(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{F}_2)),$$

which is a proper subgroup of $SL_2(\mathbb{Z})$. Descending the equivariant biholomorphism $\widetilde{\tau}$, we obtain

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \cong \mathscr{H}/\Gamma(2).$$

Many moduli spaces carry a natural Hodge theory, but some do not.

2 Hodge theory of elliptic curves

We now consider the Legendre family

$$V = \left\{ y^2 = x(x-1)(x-\lambda) \mid \lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\} \right\}.$$

Its fiber over $\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ is denoted by E^o_λ. Thus the family endows the hyperbolic space

$$\mathbb{P}^1 \setminus \{0,1,\infty\}$$

with a natural "moduli space" interpretation. Later, we will compactify this family and define it over $\mathbb{P}^1$. We denote the corresponding compactification of E^o_λ by E_λ. This compactification introduces degenerations of Hodge structures. We will also introduce the notion of *degree*; using degree, we can see why the derivative of the period map is an isomorphism.

We first discuss the **topology** of E_λ. For the projective curve E_λ, choosing a symplectic basis δ, γ gives a decomposition

$$H_1(E_\lambda, \mathbb{Z}) = \mathbb{Z}\delta \oplus \mathbb{Z}\gamma.$$

The intersection form with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This fiberwise intersection form gives the polarization of the Hodge structure and is closely related to the Hodge metric in families. The Hodge metric plays an important role in variation of Hodge structures.

Hodge theory combines three kinds of geometry: topology, differential geometry, and complex geometry. We next discuss the complex geometry of E_λ. Regarded as a Riemann surface, E_λ carries the holomorphic differential form

$$\omega_\lambda = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

Although this expression is written in one affine chart, it glues with the corresponding expressions in the other affine charts, and hence defines a global holomorphic differential form. When performing loop integration, this affine expression is sufficient, but it is useful to remember that the form is globally defined. By the basic theory of elliptic curves, the space of holomorphic 1-forms is one-dimensional, so ω_λ is unique up to scaling.

Now consider periods. By de Rham theory, a closed 1-form defines a cohomology class, equivalently a functional on homology:

$$\int_{-} \omega_\lambda : H_1(E_\lambda, \mathbb{C}) \longrightarrow \mathbb{C}, \qquad \alpha \longmapsto \int_\alpha \omega_\lambda.$$

With respect to the dual basis δ^*, γ^* of $H^1(E_\lambda, \mathbb{C})$, we have

$$[\omega_\lambda] = \delta^* \int_\delta \omega_\lambda + \gamma^* \int_\gamma \omega_\lambda.$$

We call

$$(A(\lambda), B(\lambda)) = \left(\int_\delta \omega_\lambda, \int_\gamma \omega_\lambda\right)$$

the periods of E_λ. These periods depend on the choice of ω_λ. However, as noted above, the holomorphic 1-form is unique up to scaling. Therefore the ratio

$$\tau(\lambda) = \frac{B(\lambda)}{A(\lambda)}$$

is independent of this scaling, provided $A(\lambda) \neq 0$. The fact that $A(\lambda)$ is nonzero follows from the following theorem.

Theorem 1. Let $H^{1,0}_\lambda \subset H^1(E_\lambda, \mathbb{C})$ be the subspace spanned by $[\omega_\lambda]$. Then

$$H^1(E_\lambda, \mathbb{C}) = H^{1,0}_\lambda \oplus H^{0,1}_\lambda.$$

Here the left-hand side is topological, while the decomposition on the right-hand side depends on the complex structure of E_λ.

Proof. Consider the intersection pairing

$$H^1(E_\lambda, \mathbb{C}) \times H^1(E_\lambda, \mathbb{C}) \longrightarrow \mathbb{C},$$

which is induced by the intersection form on $H_1(E_\lambda, \mathbb{C})$. If

$$A = \int_\delta \omega_\lambda, \qquad B = \int_\gamma \omega_\lambda,$$

then, up to the convention for the chosen symplectic basis,

$$\mathrm{i} \int_{E_\lambda} [\omega] \cup [\overline{\omega}] = 2\, \mathrm{Im}(B\overline{A}).$$

On the other hand, locally we can write $\omega = f\, dz$. Hence

$$\mathrm{i}\, \omega \wedge \overline{\omega} = \mathrm{i}|f|^2\, dz \wedge d\overline{z} = 2|f|^2\, dx \wedge dy,$$

where $dx \wedge dy$ is the orientation defined by the complex structure. Thus the integrand is a positive function times the volume element. Therefore

$$\mathrm{i} \int_{E_\lambda} \omega \wedge \overline{\omega} > 0,$$

and consequently

$$\mathrm{Im}(B\overline{A}) > 0.$$

In particular, $A \neq 0$ and $B \neq 0$.

We may therefore normalize ω_λ so that $A = 1$. Under this normalization,

$$\mathrm{Im}\, B > 0.$$

Suppose now that $H_\lambda^{1,0}$ and $H_\lambda^{0,1}$ do not give a direct-sum decomposition of $H^1(E_\lambda, \mathbb{C})$. Since both are one-dimensional subspaces, this would imply

$$H_\lambda^{1,0} = H_\lambda^{0,1}.$$

Hence $[\overline{\omega}] = c[\omega]$ for some $c \in \mathbb{C}$. Using the normalization $A = 1$, this gives

$$\delta^* + \overline{B}\gamma^* = c(\delta^* + B\gamma^*).$$

Comparing coefficients, we obtain $c = 1$ and $\overline{B} = B$, contradicting $\operatorname{Im} B > 0$. Therefore

$$H^1(E_\lambda, \mathbb{C}) = H_\lambda^{1,0} \oplus H_\lambda^{0,1}.$$

□

Locally on a disk

$$\Delta \subset \mathbb{P}^1 \setminus \{0, 1, \infty\},$$

the Legendre family is a smooth product family by Ehresmann's theorem. The period map

$$\Delta \longrightarrow \mathscr{H}, \qquad \lambda \longmapsto B(\lambda),$$

after the normalization $A(\lambda) = 1$, measures the variation of the complex structure, and it is holomorphic.

The definitions of the period functions A and B on a disk Δ depend on the choice of a symplectic homology basis $\{\delta, \gamma\}$. A different symplectic basis $\{\delta', \gamma'\}$ is related to the old one by

$$\begin{aligned} \delta' &= a\delta + b\gamma, \\ \gamma' &= c\delta + d\gamma, \end{aligned}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The periods with respect to the new basis are related to those with respect to the old basis by

$$\begin{aligned} A' &= aA + bB, \\ B' &= cA + dB. \end{aligned}$$

Thus the new period vector (A', B') is obtained by multiplying the old period vector (A, B) by the matrix T. Therefore the corresponding τ-invariants are related by the fractional linear transformation

$$\tau' = \frac{B'}{A'} = \frac{c + d\tau}{a + b\tau}.$$

We can quotient out this ambiguity by the $SL_2(\mathbb{Z})$-action, obtaining a well-defined holomorphic map

$$\Delta \longrightarrow \mathscr{H}/SL_2(\mathbb{Z}).$$

3 Non-constancy of the period map τ

We end this lecture by showing that the period map is nonconstant. From the moduli-space point of view this is clear: the elliptic curves vary in the family, and hence the corresponding moduli map is nonconstant. Nevertheless, we can also prove this more directly.

We show that τ is a nonconstant function of λ by studying its behavior along the ray $\lambda > 2$ on the real axis. More precisely, we will see that $\tau(\lambda)$ is asymptotically proportional to $\log \lambda$. Assume $\lambda \gg 2$. Then

$$\int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \sim \int_\delta \frac{dx}{x\sqrt{-\lambda}} = \frac{2\pi}{\sqrt{\lambda}}.$$

By deforming the path of integration, we find that

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = -2\int_1^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

The difference between the last integrand and

$$\frac{1}{x\sqrt{x-\lambda}}$$

has an asymptotically negligible integral. Thus the main contribution is the residual integral

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}}.$$

This integral can be computed explicitly:

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}} = \frac{4}{\sqrt{\lambda}} \arctan \frac{\sqrt{1-\lambda}}{\sqrt{\lambda}} \sim \frac{2\mathrm{i}}{\sqrt{\lambda}} \log \lambda.$$

Consequently,

$$\tau(\lambda) \sim \frac{\mathrm{i}}{\pi} \log \lambda.$$

In particular, $\tau(\lambda)$ is nonconstant.