

We continue with the basic example of the Legendre family of elliptic curves

$$E_\lambda := \{y^2 = x(x-1)(x-\lambda)\} \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

The main ideas in this lecture already appear in this example.

Recall from the previous lecture that the Legendre family admits a compactification over \mathbb{P}^1 with singular fibres over $\{0, 1, \infty\}$. The fibres over 0 and ∞ are semistable, while the fibre over 1 is non-reduced. To avoid parabolic structures, we pass to a semistable reduction by the double cover

$$\mathbb{P}^1 \setminus \{0, +1, -1, \infty\} \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad t \longmapsto \lambda = t^2.$$

Set

$$S := \{0, +1, -1, \infty\}.$$

After this base change, we obtain a family

$$g : E_\delta \longrightarrow \mathbb{P}^1 \setminus S$$

fitting into the commutative diagram

$$\begin{array}{ccc} E_\delta & \longrightarrow & E \\ g \downarrow & & \downarrow \\ \mathbb{P}^1 \setminus S & \longrightarrow & \mathbb{P}^1 \setminus \{0, 1, \infty\}. \end{array}$$

The family g has a semistable compactification

$$\bar{g} : \bar{E}_\delta \longrightarrow \mathbb{P}^1$$

whose singular fibres lie over S . Although one can work directly with the original non-semistable family, doing so requires keeping track of the associated parabolic structure. The semistable reduction is therefore a convenient simplification.

1 The local system and the Hodge bundle on $\mathbb{P}^1 \setminus S$

Consider the weight-one local system

$$\mathbb{V}_{\mathbb{P}^1 \setminus S} := R^1 g_* \mathbb{Q}_{E_\delta}$$

on $\mathbb{P}^1 \setminus S$. The associated flat holomorphic bundle is

$$(\mathcal{V}_{\mathbb{P}^1 \setminus S}, \nabla^{\text{GM}}) := (\mathbb{V}_{\mathbb{P}^1 \setminus S} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^1 \setminus S}, \nabla^{\text{GM}}),$$

where ∇^{GM} is the Gauss–Manin connection. Locally, the transition functions of the local system are constant, and this is precisely what gives the flat connection.

This flat bundle carries the Hodge filtration. In the present family, the Hodge line bundle is locally generated by the holomorphic differential

$$\omega_t := \frac{dx}{\sqrt{x(x-1)(x-t^2)}}.$$

Thus

$$E_{\mathbb{P}^1 \setminus S}^{1,0} := \bigcup_{t \in \mathbb{P}^1 \setminus S} \mathbb{C}\omega_t \subset \mathcal{V}_{\mathbb{P}^1 \setminus S}$$

is a holomorphic line subbundle. The corresponding period map is described by

$$\tau(t) = \mathbb{C}\omega_t \subset \mathcal{V}_t.$$

In other words, $\tau(t)$ records the Hodge line $H^{1,0}(E_t) \subset H^1(E_t, \mathbb{C})$.

2 Deligne’s canonical extension over a punctured disk

To use algebraic geometry on a compact base, we must extend the flat bundle across the boundary. This is the role of Deligne’s canonical extension.

Let Δ be the unit disk and let $\Delta^* = \Delta \setminus \{0\}$. Let \mathbb{L} be a local system on Δ^* . The associated flat bundle is

$$(\mathcal{L}, \nabla) = (\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{\Delta^*}, \nabla).$$

We want to extend this bundle over Δ in a canonical way.

Let T be the local monodromy operator around 0. More explicitly, if $\gamma \in \pi_1(\Delta^*)$ is the positive generator and ℓ is a flat multivalued section, then analytic continuation of ℓ along γ gives $T(\ell)$. If the monodromy is unipotent, then

$$M := \log T = (T - I) - \frac{(T - I)^2}{2} + \frac{(T - I)^3}{3} - \dots$$

is a finite sum. In this unipotent case, no choice of branch is involved in defining $\log T$. For general quasi-unipotent monodromy, Deligne’s extension is obtained by fixing the eigenvalues of the residue in a prescribed interval.

Define the residue operator

$$R := -\frac{1}{2\pi i} M.$$

For a flat multivalued section ℓ , set

$$\tilde{\ell} := \exp(R \log z) \ell.$$

Although ℓ and $\log z$ are individually multivalued, their combination is single-valued. Indeed, after going once around the origin,

$$\log z \mapsto \log z + 2\pi i, \quad \ell \mapsto T^{-1} \ell = \exp(-2\pi i R) \ell,$$

and hence

$$\exp(R(\log z + 2\pi i)) \exp(-2\pi i R) \ell = \exp(R \log z) \ell.$$

Definition 1 (Deligne's canonical extension). The Deligne extension $\overline{\mathcal{L}}$ is the locally free \mathcal{O}_Δ -module generated by the single-valued sections $\tilde{\ell} = \exp(R \log z) \ell$, where ℓ runs over a basis of flat multivalued sections of \mathbb{L} .

The connection extends to $\overline{\mathcal{L}}$ with logarithmic pole at the origin. Indeed, since ℓ is flat, we compute

$$\begin{aligned} d(\exp(R \log z) \ell) &= d(\exp(R \log z)) \ell + \exp(R \log z) d\ell \\ &= R \exp(R \log z) \ell \frac{dz}{z}. \end{aligned}$$

Thus

$$\nabla \tilde{\ell} = R \tilde{\ell} \frac{dz}{z},$$

so the residue of the logarithmic connection is precisely R .

3 Extension of the Hodge bundle

The subtle point is not merely extending the flat bundle, but extending the Hodge filtration subbundles. Let C be a compact Riemann surface, let $S \subset C$ be finite, and let \mathbb{L} be a local system on $C \setminus S$. On $C \setminus S$, the associated flat bundle

$$(\mathcal{L}, \nabla) = (\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{C \setminus S}, \nabla)$$

comes with Hodge filtration subbundles $F^p \mathcal{L}$.

A priori, the extension of $F^p \mathcal{L}$ across S need not be visibly algebraic. It is not automatic from the definition of a holomorphic subbundle on $C \setminus S$ that its limit inside the Deligne extension is an algebraic subbundle. Schmid's nilpotent orbit theorem gives the required control: it shows that the Hodge filtration has a well-defined limiting filtration and hence extends across the boundary in a controlled algebraic manner. Simpson's Hermitian–Yang–Mills perspective gives a different conceptual explanation: in that framework, the nilpotent orbit theorem is reflected in precise metric estimates for harmonic bundles on non-compact curves.

We now recall the local form of Schmid's theorem. Choose a punctured disk $\Delta^* \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and lift the period map to the universal cover $\mathcal{H} \rightarrow \Delta^*$, where $q = \exp(2\pi i z)$ and $z \in \mathcal{H}$. We obtain a lifted period map

$$\tilde{\tau} : \mathcal{H} \longrightarrow D,$$

where in the present example $D = \mathcal{H}$ and the compact dual is $\check{D} = \mathbb{P}^1$. If γ denotes the local monodromy, then

$$\tilde{\tau}(z + 1) = \gamma \cdot \tilde{\tau}(z).$$

Since the family is semistable, γ is unipotent. Put

$$N := \log \gamma.$$

Define

$$\tilde{\psi} : \mathcal{H} \longrightarrow \check{D}, \quad \tilde{\psi}(z) := \exp(-zN)\tilde{\tau}(z).$$

Then $\tilde{\psi}$ is invariant under $z \mapsto z + 1$. Indeed,

$$\begin{aligned} \tilde{\psi}(z+1) &= \exp(-(z+1)N)\tilde{\tau}(z+1) \\ &= \exp(-(z+1)N)\gamma\tilde{\tau}(z) \\ &= \exp(-(z+1)N)\exp(N)\tilde{\tau}(z) \\ &= \exp(-zN)\tilde{\tau}(z) = \tilde{\psi}(z). \end{aligned}$$

Therefore $\tilde{\psi}$ descends to a holomorphic map

$$\psi : \Delta^* \longrightarrow \check{D}.$$

Theorem 2 (Schmid's nilpotent orbit theorem, local form). The map $\psi : \Delta^* \rightarrow \check{D}$ extends holomorphically to

$$\psi : \Delta \longrightarrow \check{D}.$$

Let $a := \psi(0)$. Then the nilpotent orbit

$$O(z) := \exp(zN)a$$

has the following properties. There exists $C > 0$ such that:

1. if $\text{Im } z \geq C$, then $O(z) \in D$;
2. for every $\varepsilon > 0$, there exists $A(\varepsilon) > 0$ such that

$$d(O(z), \tilde{\tau}(z)) \leq A(\varepsilon) \exp(-2\pi(1-\varepsilon)\text{Im } z)$$

for all $\text{Im } z \geq C$, where d is the invariant Hodge metric on D .

This theorem explains why the Hodge line extends algebraically. First by holomorphicity of ψ , we get a holomorphic extension of Hodge line bundle on Δ . Then by GAGA on the compact Riemann surface, every holomorphic line bundle is algebraic so that we get an algebraic extension¹.

¹Remark. The original note explain the algebraicity as follows: as $q \rightarrow 0$ on Δ^* , equivalently $\text{Im } z \rightarrow +\infty$ on the universal cover \mathcal{H} (with $\exp(2\pi iz) = t$), the principal part of singularity of $\tau(t) = E_t^{1,0}$ is $\exp(\frac{1}{2\pi i}N \log t)$. By unipotent of N this is simply a polynomial in $\log t$, hence algebraic.

4 Extension of the Higgs bundle

We now pass from the variation of Hodge structure to its associated logarithmic Higgs bundle. After taking Deligne's canonical extension and extending the Hodge filtration, we obtain

$$E_{\mathbb{P}^1}^{1,0} \subset \bar{\mathcal{V}}_{\mathbb{P}^1}, \quad E_{\mathbb{P}^1}^{0,1} := \bar{\mathcal{V}}_{\mathbb{P}^1} / E_{\mathbb{P}^1}^{1,0}.$$

The logarithmic Gauss–Manin connection induces the Higgs field

$$\theta : E_{\mathbb{P}^1}^{1,0} \longrightarrow E_{\mathbb{P}^1}^{0,1} \otimes \Omega_{\mathbb{P}^1}^1(\log S).$$

Because the variation has weight one and comes from elliptic curves, the polarization identifies

$$E^{0,1} \simeq (E^{1,0})^\vee.$$

We will use the following two facts:

1. $\theta \neq 0$;
2. $\deg E^{1,0} > 0$.

The first fact follows from non-constancy of the period map. The second follows either from the Griffiths curvature formula or from the Yang–Mills–Higgs metric and polystability.

4.1 The Higgs field as the derivative of the period map

The Higgs field is the infinitesimal form of the period map. More precisely, Griffiths transversality gives

$$\nabla F^1 \subset F^0 \otimes \Omega_{\mathbb{P}^1}^1(\log S),$$

and the induced map on the quotient is

$$\theta : E_{\mathbb{P}^1}^{1,0} \longrightarrow E_{\mathbb{P}^1}^{0,1} \otimes \Omega_{\mathbb{P}^1}^1(\log S).$$

Equivalently, after contraction with tangent vectors, this gives

$$T_{\mathbb{P}^1}(-\log S) \longrightarrow \mathrm{Hom}(E^{1,0}, E^{0,1}),$$

which is precisely the differential $d\tau$ of the period map. Thus one may write, in this weight-one situation,

$$\theta = d\tau.$$

4.2 Why the two facts imply that $d\tau$ is an isomorphism

Since $|S| = 4$, we have

$$\Omega_{\mathbb{P}^1}^1(\log S) = \omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(S) \simeq \mathcal{O}_{\mathbb{P}^1}(-2 + 4) = \mathcal{O}_{\mathbb{P}^1}(2).$$

Write

$$E_{\mathbb{P}^1}^{1,0} \simeq \mathcal{O}_{\mathbb{P}^1}(d).$$

Then

$$E_{\mathbb{P}^1}^{0,1} \simeq (E_{\mathbb{P}^1}^{1,0})^\vee \simeq \mathcal{O}_{\mathbb{P}^1}(-d).$$

The Higgs field is therefore a morphism

$$\theta : \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \mathcal{O}_{\mathbb{P}^1}(2) = \mathcal{O}_{\mathbb{P}^1}(2-d),$$

or equivalently a section

$$\theta \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-2d)).$$

Since $\theta \neq 0$, we must have

$$2 - 2d \geq 0.$$

On the other hand, $\deg E^{1,0} = d > 0$, so $d \geq 1$. Hence $d = 1$.

Consequently,

$$E^{1,0} \simeq \mathcal{O}_{\mathbb{P}^1}(1), \quad E^{0,1} \otimes \Omega_{\mathbb{P}^1}^1(\log S) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \simeq \mathcal{O}_{\mathbb{P}^1}(1).$$

Thus

$$\theta : \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$$

is a nonzero endomorphism of $\mathcal{O}_{\mathbb{P}^1}(1)$. Since

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{C},$$

this morphism is multiplication by a nonzero scalar. Therefore θ is an isomorphism, and hence $d\tau$ is an isomorphism.

4.3 Proof of $\theta \neq 0$ and $\deg E^{1,0} > 0$

First, $\theta \neq 0$ because $\theta = d\tau$ and the period map τ is not constant.

It remains to show that $\deg E^{1,0} > 0$. Consider the logarithmic Higgs bundle

$$(E, \theta) := (E^{1,0} \oplus E^{0,1}, \theta).$$

Since the Higgs field has only the component

$$\theta : E^{1,0} \rightarrow E^{0,1} \otimes \Omega_{\mathbb{P}^1}^1(\log S)$$

and is zero on $E^{0,1}$, we have a Higgs subbundle

$$(E^{0,1}, 0) \hookrightarrow (E^{1,0} \oplus E^{0,1}, \theta).$$

The Hodge metric makes (E, θ) a polystable Higgs bundle with vanishing first Chern class; equivalently,

$$\deg(E) = \deg E^{1,0} + \deg E^{0,1} = 0.$$

By semistability of (E, θ) , every Higgs subbundle has slope at most $\mu(E) = 0$. Applying this to $(E^{0,1}, 0)$ gives

$$\deg E^{0,1} \leq 0.$$

Since $\deg E^{1,0} + \deg E^{0,1} = 0$, it follows that

$$\deg E^{1,0} \geq 0.$$

We now exclude equality.

Suppose $\deg E^{0,1} = 0$. Then $(E^{0,1}, 0)$ has the same slope as the polystable Higgs bundle (E, θ) . Hence it splits as a direct summand:

$$(E, \theta) \simeq (E^{0,1}, 0) \oplus (Q, \theta_Q),$$

where

$$Q = E/E^{0,1} \simeq E^{1,0}.$$

The induced Higgs field on the quotient is nilpotent. Since Q has rank one, any nilpotent endomorphism-valued one-form on Q must be zero; equivalently, a rank-one Higgs field which is nilpotent is identically zero. Therefore

$$\theta_Q = 0.$$

The above splitting then forces the original Higgs field to be zero, contradicting $\theta = d\tau \neq 0$. Hence

$$\deg E^{0,1} < 0,$$

and therefore

$$\deg E^{1,0} > 0.$$

Combining this with the previous subsection gives

$$E^{1,0} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$$

and shows that the derivative of the period map is an isomorphism. In the next lecture, we will recover this result from the viewpoint of moduli theory and identify

$$\rho(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})) = \Gamma(2).$$