Hodge Theory Course Notes Summer 2026

Lecture 4 — 2026-06-11

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We will continue the discussion of the example of the Legendre family of elliptic curves

$$\{y^2 = x(x-1)(x-\lambda)\} \to \mathbb{P}^1 \setminus \{0,1,\infty\}.$$

We will see that all the ideas come from this example. Recall that last time, we showed the Legendre family

$$\{y^2 = x(x-1)(x-\lambda)\} \to \mathbb{P}^1 \setminus \{0,1,\infty\}$$

has the compactification over $\mathbb{P}^1$ which turns out to have three singular fibers $\{0,1,\infty\}$, with semistable fiber at $\{0,\infty\}$ and non-reduced fiber over $\{1\}$. To simplify the situation, we can take a semistable reduction, via the double cover $\mathbb{P}^1 \setminus \{0,+1,-1,\infty\} \to \mathbb{P}^1 \setminus \{0,1,\infty\}$, and let us denote $S = \{0,+1,-1,\infty\}$, and hence we have the following commutative diagram

$$\begin{array}{ccc} E_\delta & \longrightarrow & E \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ \mathbb{P}^1 \setminus S & \longrightarrow & \mathbb{P}^1 \setminus \{0,1,\infty\} \end{array}$$

The family g has a semistable compactification $\bar{E}_\delta \to \mathbb{P}^1$ with singular fiber over S and denote the indeterminacy locus of $\bar{g}$ as $\Delta \subset \bar{E}_\delta$. It is worth mentioning that we can also work with non semistable family, but we need to deal with the parabolic structure, and doing semistable reduction is merely for simplicity.

1 Local system and Hodge bundle on $\mathbb{P}^1 \setminus S$

Consider the local system

$$\mathbb{V}_{\mathbb{P}^1} = R^1 g_* \mathbb{Q}_{\bar{E}_\delta \setminus \Delta},$$

we can define the deRham bundle using this local system

$$\mathcal{V}_{\mathbb{P}^1\setminus S} = (\mathbb{V}_{\mathbb{P}^1\setminus S} \otimes \mathcal{O}_{\mathbb{P}^1\setminus S}, \nabla^{GM}),$$

where the Gauss–Manin connection is globally defined since the transition function for local system are constant functions. This flat holomorphic bundle admits a Hodge filtration bundle

$$\bigcup_{t\in\mathbb{P}^1\setminus S} \mathbb{C}\frac{dx}{\sqrt{x(x-1)(x-t^2)}} \subset \mathcal{V}_{\mathbb{P}^1\setminus S},$$

which is a holomorphic sub line bundle.

Recall that we defined the period map as

$\tau(t) = \mathbb{C}\frac{dx}{\sqrt{x(x-1)(x-t^2)}} \subset \mathcal{V}_{\mathbb{P}^1\setminus S}$.

2 Deligne's canonical extension over punctured disk

To do algebraic geometry, we need to take a compactification, even in log geometry. We now introduce Deligne's canonical extension.

Let Δ be the unit disk. Let $\mathbb{L}$ be a local system on Δ^*. As usual we can define the deRham bundle $(\mathcal{L} = \mathbb{L} \otimes \mathcal{O}_{\Delta^*}, \nabla^{GM})$, we try to find the canonical extension of this de Rham bundle over Δ.

To do this, we define

$$M = \log T,$$

where T_γ is the local monodromy of $\mathbb{L}$ around loop $\gamma \in \pi_1(C \setminus S, \{*\})$. Here the local monodromy operator $T \in \mathrm{Aut}(\mathbb{L}|_{\{*\}})$ is defined as follows: let $\ell \in \mathbb{L}|_*$ be a vector on $\mathbb{L}|_*$ then we can do parallel transport (analytic continuation) around the loop γ, and come back to $T(\ell)$. One crucial point we need to mention is that $\log T_\gamma$ is not well-defined: it is multivalued and requires a branch cut. However when local monodromy is unipotent, then

$$\log T = (T - I) - \frac{(T-I)^2}{2} + \frac{(T-I)^3}{3} - \cdots ,$$

is actually finite sum and no branch cut is involved. For Deligne's canonical extension, one fixes this ambiguity by choosing the eigenvalues of the residue in a specified interval.

We define the residue operator to be $R = -\frac{1}{2\pi i} M$ We then collect all the section of the form

$$\tilde{\ell} := e^{R \log z} \cdot \ell,$$

note that ℓ is multivalued flat section, while $\tilde{\ell}$ is single-valued but not necessarily flat. To the single-valued of $\tilde{\ell}$, assume that we loop around 0 then we have

$$\log z \mapsto \log z + 2\pi i$$

and

$$\ell \mapsto e^{-2\pi i R} \ell,$$

so combining them together, we see that it is single-valued!

We can now define the Deligne's canonical extension, let $\bar{\mathcal{L}}$ be the sheaf generated by all the above sections which gives an extension of the deRham bundle $(\mathcal{L}, \nabla)$. The connection of the extension has logarithmic pole, let us compute it

$$d\left(e^{R \log z} \ell\right) = d\left(e^{R \log z}\right) \ell + e^{R \log z} d\ell.$$

for a flat section ℓ we have second term is 0 and hence

$$d\left(e^{R \log z} \ell\right) = R e^{R \log z} \ell \frac{dz}{z},$$

that is

$$d\tilde{\ell} = R \frac{dz}{z} \tilde{\ell},$$

and hence we have the logarithmic pole.

3 Extension of Hodge bundle

The crucial point is the extension of the Hodge filtration bundles $F^p\mathcal{V}$, which is highly nontrivial. Let C be a compact Riemann surface, let $S \subset C$ be a finite set of points, and let $\mathbb{L}$ be a local system on $C \setminus S$. Consider the de Rham bundle

$$(\mathcal{L}, \nabla) = \left(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_{C\setminus S}, \nabla\right)$$

together with its Hodge filtration subbundles $F^p\mathcal{L}$, defined on $C \setminus S$. A priori, the extension of these Hodge bundles across S need not be algebraic; in principle, it could be a transcendental object. The point is that Schmid's nilpotent orbit theorem shows that these extended Hodge bundles are in fact well-behaved algebraic objects. Simpson's Hermitian–Yang–Mills approach gives a different and more conceptual perspective: in that framework, Schmid's nilpotent orbit theorem can be viewed essentially as a metric estimate (cf. Simpson's paper on harmonic maps on non-compact curves).

Let us talk about Schmid's nilpotent orbit theorem. First we have the period map $\mathbb{P}^1 \setminus \{0, 1, \infty\} \to \mathscr{H}/\Gamma(2)$, and we can pick a punctured disk $\Delta^* \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$, then we have the induced map

$$\Delta^* \to \mathscr{H}/\rho(\pi_1(\Delta^*))$$

And hence we have the commutative diagram

$$\begin{array}{ccc} \pi_1(\Delta^*) \circlearrowright \mathscr{H} & \xrightarrow{\tilde{\tau}} & D = \mathscr{H} \circlearrowleft \gamma \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow[\tau]{} & D/\Gamma \end{array}$$

where the period domain D happens to be the upper half space. Here the universal cover $\mathscr{H} \to \Delta^*$ carries a natural $\pi_1(\Delta^*)$ action while the period domain D carries a monodromy action Γ. Note that the central fiber is semistable, so that the monodromy action is unipotent, in general this is quasi-unipotent (see Borel's monodromy theorem). We have the equivariant relation

$$\tilde{\tau}(z + 1) = \gamma \cdot \tilde{\tau}(z).$$

We then define

$$\widetilde{\psi} : \mathscr{H} \to \check{D}, \quad z \mapsto \exp(-zN)\tilde{\tau}(z),$$

here $N = \log \gamma = (\gamma - I) - \frac{(\gamma-I)^2}{2} + \ldots + (-1)l\frac{(\gamma-I)^l}{l}$ (which is called normalization operator), note that the target is the compact dual $\check{D} = \mathbb{P}^1$ of the period domain $D = \mathscr{H}$ since multiplying $\exp(-zN)$ may break the polarization. Then one can check

$$\tilde{\psi}(z + 1) = \tilde{\psi}(z).$$

Indeed

$$\begin{aligned} \widetilde{\psi}(z + 1) &= \exp(-(z + 1)N)\widetilde{\tau}(z + 1) \\ &= \exp(-(z + 1)N)\gamma\widetilde{\tau}(z) \\ &= \exp(-(z + 1)N) \exp(N)\widetilde{\tau}(z) \end{aligned}$$

Since N commutes with itself, we have

$$\exp(-(z+1)N)\exp(N) = \exp(-zN - N)\exp(N) = \exp(-zN).$$

Hence it descends to

$$\psi(z) : \Delta^* \to \check{D} = \mathbb{P}^1,$$

Schmid's nilpotent orbit theorem tells us the asymptotic behavior of this period map:

Theorem 1 (Nilpotent orbit theorem)**.** ψ is continuous, and holomorphic over Δ^*, and $a = \psi(0) \in \mathbb{P}^1$, which extends to a holomorphic mapping

$$\psi(z) : \Delta \to \mathbb{P}^1.$$

Moreover, consider the nilpotent orbit $O(z) = \exp(zN) \cdot a$ for $z \in \mathscr{H}$, there exists a real constant $C > 0$ such that

(a) For $\mathrm{Im}(z) \geq C$ we have $O(z) \in D = \mathscr{H}$,

(b) given $\epsilon > 0$, then

$$d(O(z), \tilde{\tau}(z)) \leq A(\epsilon)\exp(-2\pi(1-\epsilon)\mathrm{Im}z),$$

for $\mathrm{Im}(z) \geq C$, for some constant $A(\epsilon) > 0$. Here $d(\cdot,\cdot)$ denotes the natural invariant metric on the period domain D called Hodge metric.

Now we come back to our problem, we want to show the extension of the Hodge filtration bundle is algebraic. Note that when $t \in \Delta^*$ goes to 0, then $z \in \mathscr{H}$ on the universal cover has $\mathrm{Im}(z) \to +\infty$. The principal part of singularity of $\tau(t) = E_t^{1,0} \subset \mathcal{V}_t$ the Hodge line bundle determined by τ, is $\exp(\frac{1}{2\pi i}M\log t)$ and $\tau(t)$, with nilpotent correction $N(t)$ is algebraic in $\bar{\mathcal{V}}|_\Delta$. Thus there exists an extension of $E_\Delta^{1,0} \subset \mathcal{V}|_\Delta$.

4 Extension of Higgs bundle

Now we consider the extension of Higgs bundle. Finally, we want to prove derivative of the period map is an isomorphism.

Now we have the extension of Hodge bundle $E_{\mathbb{P}^1}^{1,0} \subset \mathcal{V}_{\mathbb{P}^1}$ with the holomorphic quotient $\mathcal{V}/E_{\mathbb{P}^1}^{1,0} = E_{\mathbb{P}^1}^{0,1}$. And we have the logarithmic Higgs field

$$\theta : E_{\mathbb{P}^1}^{1,0} \to E_{\mathbb{P}^1}^{0,1} \otimes \Omega_{\mathbb{P}^1}^1(\log S)$$

such that $(E^{0,1})^\vee = E^{1,0}$. We will show that

(1) $\theta \neq 0$ (recall that we proved before that $\tau(t)$ is not constant, we will see this implies that $\theta \neq 0$),

(2) $\deg E^{1,0} > 0$ (this is an easy consequence of Yang-Mills-Higgs metric or Griffiths curvature formula).

A side remark, this is closely related to the derivative of the period mapping. This is a general fact that Hodge theory is related to the period mapping via Higgs field θ. Higgs bundle is much more useful than vector bundle, as when $\theta \neq 0$ we can connect topology and geometry. When doing algebraic geometric problem we typically require $\theta \neq 0$.

4.1 Higgs field, Kodaira-Spencer map and derivative of period mapping

We show that $\theta = d\tau$ that is Higgs field is actually the derivative of the period mapping.

First consider the Higgs field

$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_{\mathbb{P}^1}(\log S),$$

and we can contract θ with a. On the other hand, we have the derivative of the period mapping

$$d\tau : T(\mathbb{P}^1 \setminus S) \cong T_{\mathbb{P}^1}(\log S) \to TD = T(\mathscr{H}/\Gamma(2)) \subset \mathrm{Hom}(E^{1,0}, E^{0,1}),$$

up to dual we see that $\theta = d\tau$. This is a general phenomenon, that we will give a rigorous proof later.

4.2 Proof of (1)+(2) implies isomorphism of derivative of period mapping

Let us first see why (1) and (2) above imply that the derivative of the period mapping is isomorphism. Note that since all the line bundle on $\mathbb{P}^1$ are of the form $\mathcal{O}(d)$, if we write

$$\Omega^1_{\mathbb{P}^1}(\log S) = \omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(S) = \mathcal{O}_{\mathbb{P}^1}(-2+4) = \mathcal{O}_{\mathbb{P}^1}(2).$$

We then assume that $L = E^{1,0}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(d)$ and hence $\deg E^{1,0} = d$. Then the Higgs field θ is equivalent to a section

$$\theta \in H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(2-2d)).$$

By assumption, we have $\theta \neq 0$ and therefore

$$2 - 2d \geq 0.$$

On the other hand, we have $\deg E^{1,0} = d \geq 1$ and then the only possible choice is $d = 1$ and hence

$$\theta : \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)$$

This map has to be an isomorphism as $0 \neq \theta \in \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ and hence $\theta = \lambda \mathrm{Id}$ for $\lambda \neq 0$. And hence the derivative of the period domain has to be isomorphism.

4.3 Proof of (1) and (2)

Now let us prove (1) and (2) holds. First, for (1), we already showed that $\theta = d\tau$ and we know that τ is not constant, and hence this implies that $\theta \neq 0$.

Next we show that $\deg E^{1,0} > 0$. We have

$$\theta = d\tau : T_{\mathbb{P}^1}(-\log S) \to (E^{1,0})^\vee \otimes E^{0,1} = (E^{0,1})^{\otimes 2},$$

We have

$$(E^{0,1}, 0) \hookrightarrow (E^{1,0} \oplus E^{0,1}, \theta)$$

is subHiggs bundle (This is subHiggs bundle because $\theta\left(E^{0,1}\right) = 0 \subset E^{0,1} \otimes \Omega^1_{\mathbb{P}^1}(\log S)$).

The key point is $(E^{1,0} \oplus E^{0,1}, \theta)$ admits Hodge metric and hence it's polystable with vanishing first Chern class (by Kobayashi–Hitchin correspondence for Higgs bundle)! And hence

$$\deg(E^{0,1} \oplus E^{1,0}) = 0$$

And hence by stability condition we have $\deg E^{0,1} \leq \deg(E^{0,1} \oplus E^{1,0})/\mathrm{rk}(E^{0,1} \oplus E^{1,0}) = 0$.

We then show it cannot be zero. When $\deg E^{0,1} = 0$, by stability condition, this implies that we have a splitting

$$(E, \theta) \simeq \left(E^{0,1}, 0\right) \oplus (Q, \theta_Q),$$

with $Q = E/E^{0,1} \simeq E^{1,0}$, by nilpotency of the induced Higgs field, we have $\theta_Q = 0$. (Indeed this is a general fact that rank-one Higgs bundle with nilpotent Higgs field has zero Higgs field). Hence we have the splitting

$$(E^{1,0}, 0), (E^{0,1}, 0)$$

and hence we deduce that $\theta = 0$ but $\theta = d\tau = 0$, but τ is not constant a contradiction.

Next time, we will recover this result using moduli theory. And show that $\rho(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})) = \Gamma(2)$.