Hodge Theory Course Notes Summer 2026

Lecture 5 — 2026-06-15

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These notes discuss the Gauss–Manin connection and two constructions of graded Higgs bundles arising from geometry. The first is the classical system of Hodge bundles attached to a variation of Hodge structure. The second is the graded Higgs bundle arising from the Kodaira–Spencer map, in the spirit of Viehweg–Zuo. We also explain Katz–Oda's construction of Gauss-Manin connection.

1 Overview

Let $Y = U \cup S$, where U is quasi-projective and S is SNC divisor, and let $f : X \to Y$ be a proper morphism which is smooth over $U = Y \setminus S$. Let Δ be the discriminant locus of f, for technical reasons, we also assume that $\Delta \to S$ is smooth.

We want to study two types of Higgs bundles attached to this family:

(1) The classical one is the system of Hodge bundles attached to this family (due to Griffiths/Simpson). Griffiths introduced the variation of Hodge structure, it carries a Gauss–Manin connection which is not a linear operator. In Algebraic geometery, we do not like the non-linear operator. At that time people do not have good idea, but in Simpson's thesis, we find just taking grading of variation of Hodge structure, we get linear operator.

We may worry that if we take the grading, we may lose information. For example, suppose we have a filtered holomorphic vector bundle

$$0 \subset F^1 \subset F^0 = V.$$

Taking the associated graded gives

$$\operatorname{gr}_F V = F^1 \oplus V/F^1.$$

But we forget the extension class of the exact sequence

$$0 \to F^1 \to V \to V/F^1 \to 0.$$

But globally, you can use this linear object to recover the non-linear object, which is the interesting part of the non-Abelian Hodge correspondence.

This linear object is very useful, e.g. Higgs field is the derivative of the period map and many hyperbolicity can be realized by the Higgs bundle. However it has some disadvantage, given a family, it can happen that Hodge structure is constant (i.e. variation of Hodge structure is trivial) while the complex structure varies. And we introduce the second type of Higgs bundle.

(2) Graded Higgs bundle as extension of classical Kodaira-Spencer map. Kodaira Spencer map faithful measure the variation of complex structure. The disadvantage is that it does not directly coming from the topology. The advantage is that we can get some complex hyperbolicity results.

We will see later the difference of the construction, as taking dual is not compatible with cohomology operator.

2 First construction: the system of Hodge bundles

Consider the logarithmic fundamental exact sequence

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0.$$

Taking exterior powers gives a natural filtration on $\Omega^p_X(\log \Delta)$. The relevant graded piece gives an exact sequence

$$0 \to f^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{X/Y}(\log \Delta) \to \mathrm{gr}^1\, \Omega^p_X(\log \Delta) \to \Omega^p_{X/Y}(\log \Delta) \to 0.$$

Here

$$\mathrm{gr}^1\, \Omega^p_X(\log \Delta) = \frac{\Omega^p_X(\log \Delta)}{f^*\Omega^2_Y(\log S) \wedge \Omega^{p-2}_X(\log \Delta)}.$$

Applying $R^q f_*$ and using the projection formula gives a connecting homomorphism

$$\theta^{p,q} \colon R^q f_* \Omega^p_{X/Y}(\log \Delta) \to \Omega^1_Y(\log S) \otimes R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log \Delta).$$

Define

$$E^{p,q} := R^q f_* \Omega^p_{X/Y}(\log \Delta).$$

For fixed weight k, the system of Hodge bundles is

$$\boxed{(E, \theta) = \left(\bigoplus_{p+q=k} E^{p,q}, \bigoplus_{p+q=k} \theta^{p,q} \right).}$$

Thus

$$\theta^{p,q} \colon E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S).$$

The connecting homomorphism may also be expressed through the Kodaira–Spencer map. There is a diagram

$$\begin{array}{ccc}
T_Y(-\log S) \otimes R^q f_* \Omega^p_{X/Y}(\log \Delta) & \xrightarrow{\theta^{p,q}} & R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log \Delta) \\
\downarrow{\scriptstyle \tau \otimes \mathrm{id}} & \nearrow{\scriptstyle \cup} & \\
R^1 f_* T_{X/Y}(-\log \Delta) \otimes R^q f_* \Omega^p_{X/Y}(\log \Delta) & &
\end{array}$$

where

$$\tau\colon T_Y(-\log S) \to R^1 f_* T_{X/Y}(-\log \Delta)$$

is the logarithmic Kodaira–Spencer map. Hence $\theta^{p,q}$ is not an abstract morphism: it is obtained by composing the Kodaira–Spencer map with the cup product.

The Higgs field can be regarded as a morphism

$$\theta\colon T_Y(-\log S) \to \mathrm{End}(E).$$

It induces a Higgs field on $\mathrm{End}(E)$,

$$\mathrm{ad}\,\theta\colon \ \mathrm{End}(E) \to \mathrm{End}(E) \otimes \Omega^1_Y(\log S), \qquad \varphi \mapsto \theta \circ \varphi - \varphi \circ \theta.$$

More precisely, the formula is interpreted using the wedge product with logarithmic forms. Since the Higgs field satisfies

$$\theta \wedge \theta = 0,$$

we have $[\theta_v, \theta_w] = 0$ for local logarithmic vector fields $v, w \in T_Y(-\log S)$. Therefore

$$\theta\big(T_Y(-\log S)\big) \subseteq \ker(\mathrm{ad}\,\theta).$$

This inclusion is important in hyperbolicity arguments. Under the Hodge metric, kernels of Higgs fields carry seminegativity properties; one can then transfer this negativity to subsheaves related to $T_Y(-\log S)$.

Using the inclusion

$$\mathrm{Im}(\theta) \subseteq \ker(\mathrm{ad}\,\theta),$$

one obtains a morphism of complexes

$$T_Y(-\log S)[0] \to \Big[\mathrm{End}(E) \xrightarrow{\mathrm{ad}\,\theta} \mathrm{End}(E) \otimes \Omega^1_Y(\log S) \xrightarrow{\mathrm{ad}\,\theta} \mathrm{End}(E) \otimes \Omega^2_Y(\log S) \to \cdots\Big].$$

Passing to hypercohomology gives a map

$$H^1\big(Y, T_Y(-\log S)\big) \to \mathbb{H}^1_{\mathrm{Higgs}}\big(Y, \mathrm{End}(E)\big).$$

The target is naturally interpreted as the tangent space to a Dolbeault moduli space at the Higgs bundle (E, θ). If Y itself varies in a family over a base B, then the ordinary Kodaira–Spencer map

$$T_0 B \to H^1\big(Y, T_Y(-\log S)\big)$$

can be composed with the above map. The resulting map

$$T_0 B \to \mathbb{H}^1_{\mathrm{Higgs}}\big(Y, \mathrm{End}(E)\big)$$

is often called the non-abelian Kodaira–Spencer map, which can be used to solved Enualt-Kerr conjecture.

3 Second construction: the graded Higgs bundle from the Kodaira–Spencer map

The local Torelli problem asks whether the infinitesimal period map is injective. In many situations the infinitesimal period map has a kernel, or even vanishes, so the classical Higgs field does not see all deformations of the complex structure. This motivates the second construction.

The second construction is the graded Higgs bundle arising directly from the Kodaira–Spencer map. It does not refer directly to a variation of Hodge structure. Its starting point is the logarithmic tangent exact sequence

$$0 \to T_{X/Y}(-\log \Delta) \to T_X(-\log \Delta) \to f^*T_Y(-\log S) \to 0.$$

Taking exterior powers gives an exact sequence

$$0 \to \bigwedge^p T_{X/Y}(-\log \Delta) \to \mathrm{gr}^0 \bigwedge^p T_X(-\log \Delta) \to \bigwedge^{p-1} T_{X/Y}(-\log \Delta) \otimes f^*T_Y(-\log S) \to 0.$$

For brevity write

$$T^p_{X/Y}(-\log \Delta) := \bigwedge^p T_{X/Y}(-\log \Delta).$$

The connecting homomorphism gives maps

$$\tau^p \colon R^{p-1} f_* T^{p-1}_{X/Y}(-\log \Delta) \otimes T_Y(-\log S) \to R^p f_* T^p_{X/Y}(-\log \Delta).$$

Equivalently, if

$$G^p := R^p f_* T^p_{X/Y}(-\log \Delta),$$

then the maps τ^p define a Higgs field

$$\tau^p \colon G^{p-1} \to G^p \otimes \Omega^1_Y(\log S).$$

For $p = 1$ this is exactly the logarithmic Kodaira–Spencer map

$$\tau^1 \colon T_Y(-\log S) \to R^1 f_* T_{X/Y}(-\log \Delta).$$

If the family is infinitesimally non-isotrivial, then τ will never be zero!

This construction is algebraic. To make it useful in geometry, one needs positivity or negativity results, usually obtained by Hodge-theoretic or analytic methods.

Proposition 1 (Negativity properties)**.** Assume the family satisfies the standard hypotheses under which the Hodge metric and the Viehweg–Zuo construction apply. Then:

(1) $\ker \theta \leq 0$ is semi-negative;

(2) $\ker \tau \leq 0$ is semi-negative and $\ker \tau < 0$ when the family has maximal variation.

Remark 2. These negativity statements do not follow from formal homological algebra alone. They require some transcendental approach.

4 The system of Hodge bundles as the associated graded of the filtered de Rham bundle

We now explain how the system of Hodge bundles is obtained as the associated graded object of the filtered de Rham bundle. The key input is E_1-degeneration of the Hodge-to-de Rham spectral sequence.

For simplicity, we omit the log structure. The fundamental exact sequence gives

$$0 \to f^*\Omega^1_Y \otimes \Omega^{p-1}_{X/Y} \to \operatorname{gr}^1 \Omega^p_X \to \Omega^p_{X/Y} \to 0.$$

In terms of complexes, this becomes

$$0 \to f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1] \to \operatorname{gr}^1 \Omega^\bullet_X \to \Omega^\bullet_{X/Y} \to 0.$$

Taking derived direct images gives a long exact sequence

$$\cdots \to \mathbb{R}^k f_* \operatorname{gr}^1 \Omega^\bullet_X \to \mathbb{R}^k f_* \Omega^\bullet_{X/Y} \xrightarrow{\partial} \mathbb{R}^{k+1} f_*\big(f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1]\big) \to \cdots .$$

By the projection formula,

$$\mathbb{R}^{k+1} f_*\big(f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}[-1]\big) \simeq \Omega^1_Y \otimes \mathbb{R}^k f_* \Omega^\bullet_{X/Y}.$$

Thus the connecting homomorphism is the Gauss–Manin connection

$$\nabla^{\mathrm{GM}} \colon \mathcal{V}^k \to \mathcal{V}^k \otimes \Omega^1_Y, \qquad \mathcal{V}^k := \mathbb{R}^k f_* \Omega^\bullet_{X/Y}.$$

In the logarithmic setting the same construction gives

$$\nabla^{\mathrm{GM}} \colon \mathcal{V}^k \to \mathcal{V}^k \otimes \Omega^1_Y(\log S).$$

Consider the truncation complex

$$\sigma^{\geq p}\Omega^\bullet_{X/Y} := [\cdots \to 0 \to \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y} \to \cdots].$$

Define the Hodge filtration bundle by

$$F^p\mathcal{V}^k := \operatorname{Im}\left[\mathbb{R}^k f_* \sigma^{\geq p}\Omega^\bullet_{X/Y} \to \mathbb{R}^k f_* \Omega^\bullet_{X/Y}\right].$$

This gives a decreasing filtration

$$\mathcal{V}^k = F^0\mathcal{V}^k \supset F^1\mathcal{V}^k \supset \cdots \supset F^{k+1}\mathcal{V}^k = 0.$$

Griffiths transversality says

$$\nabla^{\mathrm{GM}}(F^p\mathcal{V}^k) \subset F^{p-1}\mathcal{V}^k \otimes \Omega^1_Y(\log S).$$

Therefore the associated graded object carries a Higgs field

$$\theta \colon \operatorname{gr}^p_F \mathcal{V}^k \to \operatorname{gr}^{p-1}_F \mathcal{V}^k \otimes \Omega^1_Y(\log S),$$

where

$$\operatorname{gr}_F^p \mathcal{V}^k := F^p\mathcal{V}^k / F^{p+1}\mathcal{V}^k.$$

Thus we constructed a graded Higgs bundle

$$\boxed{\operatorname{gr}_{F^\bullet}(\mathcal{V}^k, \nabla^{\mathrm{GM}}) = \left(\bigoplus_{p=0}^{k} F^p\mathcal{V}^k / F^{p+1}\mathcal{V}^k, \bigoplus_{p=0}^{k} \theta_p \right).}$$

The crucial point is to identify the graded pieces $F^p\mathcal{V}^k / F^{p+1}\mathcal{V}^k$ with the Hodge bundles $R^{k-p} f_* \Omega^p_{X/Y}$. This is where E_1-degeneration enters.

Proposition 3. Let

$$E_1^{p,q} = R^q f_* \Omega^p_{X/Y} \Longrightarrow \mathbb{R}^{p+q} f_* \Omega^\bullet_{X/Y} = \mathcal{V}^{p+q}$$

be the Hodge-to-de Rham spectral sequence. Then:

(1) In general,

$$\operatorname{rk} \mathcal{V}^k \leq \sum_{p+q=k} \operatorname{rk} R^q f_* \Omega^p_{X/Y}.$$

(2) If equality holds, equivalently if the spectral sequence degenerates at E_1 in total degree k, then

$$F^p\mathcal{V}^k / F^{p+1}\mathcal{V}^k \simeq R^{k-p} f_* \Omega^p_{X/Y}.$$

Moreover, the natural map

$$\mathbb{R}^k f_* \sigma^{\geq p} \Omega^\bullet_{X/Y} \to F^p\mathcal{V}^k$$

is an isomorphism, and hence

$$\mathbb{R}^k f_* \sigma^{\geq p} \Omega^\bullet_{X/Y} \to \mathbb{R}^k f_* \Omega^\bullet_{X/Y}$$

is injective with image $F^p\mathcal{V}^k$.

Proof. For a spectral sequence of locally free sheaves, the abutment carries a filtration whose graded pieces are $E_\infty^{p,q}$. Hence

$$\operatorname{rk} \mathcal{V}^k = \sum_{p+q=k} \operatorname{rk} E_\infty^{p,q} \leq \sum_{p+q=k} \operatorname{rk} E_1^{p,q} = \sum_{p+q=k} \operatorname{rk} R^q f_* \Omega^p_{X/Y}.$$

This proves (1).

Assume equality holds. Since each $E_r^{p,q}$ is obtained from $E_1^{p,q}$ by taking successive kernels and quotients, we have

$$\operatorname{rk} E_\infty^{p,q} \leq \operatorname{rk} E_1^{p,q}.$$

The equality of the sums in total degree k therefore forces

$$\operatorname{rk} E_\infty^{p,q} = \operatorname{rk} E_1^{p,q} \qquad \text{for all } p+q=k.$$

Thus no rank is lost along the spectral sequence in total degree k. Equivalently, the spectral sequence degenerates at E_1 in this total degree. Therefore

$$E_\infty^{p,k-p} = \mathrm{gr}_F^p \mathcal{V}^k \simeq E_1^{p,k-p} = R^{k-p} f_* \Omega^p_{X/Y}.$$

It remains to justify the injectivity statement for the truncated complex. The truncated complex has a spectral sequence

$${}^{(p)}E_1^{a,b} = \begin{cases} R^b f_* \Omega^a_{X/Y}, & a \geq p, \\ 0, & a < p, \end{cases} \Longrightarrow \mathbb{R}^{a+b} f_* \sigma^{\geq p} \Omega^\bullet_{X/Y}.$$

Since the original spectral sequence degenerates at E_1 in total degree k, the truncated one also degenerates in total degree k. Hence

$$\mathrm{rk}\, \mathbb{R}^k f_* \sigma^{\geq p} \Omega^\bullet_{X/Y} = \sum_{\substack{a+b=k \\ a \geq p}} \mathrm{rk}\, R^b f_* \Omega^a_{X/Y}.$$

On the other hand,

$$\mathrm{rk}\, F^p \mathcal{V}^k = \sum_{a \geq p} \mathrm{rk}\, \mathrm{gr}_F^a \mathcal{V}^k = \sum_{\substack{a+b=k \\ a \geq p}} \mathrm{rk}\, R^b f_* \Omega^a_{X/Y}.$$

By definition the map

$$\mathbb{R}^k f_* \sigma^{\geq p} \Omega^\bullet_{X/Y} \to F^p \mathcal{V}^k$$

is surjective. Since the two sides have the same rank and are locally free under the present hypotheses, it is an isomorphism. This proves the desired injectivity into $\mathcal{V}^k$. □

5 A complex analytic proof of E_1-degeneration

We now recall the standard complex analytic proof of E_1-degeneration for a smooth proper Kähler morphism. For simplicity, assume that $f\colon X \to Y$ is smooth and proper, with compact Kähler fibers.

Proposition 4 (E_1-degeneration)**.** For a smooth proper Kähler morphism $f\colon X \to Y$, the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = R^q f_* \Omega^p_{X/Y} \Longrightarrow \mathbb{R}^{p+q} f_* \Omega^\bullet_{X/Y}$$

degenerates at E_1.

Proof. For each $s \in Y$, Hodge theory on the compact Kähler manifold X_s gives the Hodge decomposition

$$H^n_{\mathrm{dR}}(X_s, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X_s, \Omega^p_{X_s}).$$

Therefore

$$b_n(X_s) = \sum_{p+q=n} h^{p,q}(X_s).$$

The Betti numbers $b_n(X_s)$ are locally constant in a smooth proper family, because the fibers are locally topologically trivial by Ehresmann's theorem. The functions

$$s \mapsto h^{p,q}(X_s)$$

are upper semicontinuous. Since their sum is locally constant and, for compact Kähler manifolds, Hodge symmetry and Hodge decomposition identify these dimensions with the ranks of the Hodge pieces, the Hodge numbers are locally constant in a smooth proper Kähler family. By Grauert's theorem, the sheaves

$$R^q f_* \Omega^p_{X/Y}$$

are locally free and their formation commutes with base change. Moreover,

$$\left(R^q f_* \Omega^p_{X/Y}\right) \otimes \kappa(s) \simeq H^q(X_s, \Omega^p_{X_s}).$$

Similarly,

$$\mathcal{V}^n := \mathbb{R}^n f_* \Omega^\bullet_{X/Y}$$

is the holomorphic vector bundle associated to the flat local system $R^n f_* \mathbb{C}$, and

$$\operatorname{rk} \mathcal{V}^n = b_n(X_s).$$

Thus

$$\operatorname{rk} \mathcal{V}^n = b_n(X_s) = \sum_{p+q=n} h^{p,q}(X_s) = \sum_{p+q=n} \operatorname{rk} R^q f_* \Omega^p_{X/Y}.$$

By the rank criterion in the previous proposition, the Hodge-to-de Rham spectral sequence degenerates at E_1. $\square$