Hodge Theory Course Notes Summer 2026

Lecture 4 — 2026-06-15

Prof. Kang Zuo *Scribe: Yi Li*

(We omit the discussion of the Legendre family and the uniformization of elliptic curves.)

1 Kodaira–Spencer map, Hodge theory, and Higgs bundles

Given a variety, we have a complex structure. The complex structure is non-linear, and we hope to find a linear model and see the Hodge structure in the linear model, called the Hodge decomposition. We then study how it varies.

Let $f : X \to Y$ be a smooth family of (quasi)-compact complex manifolds. Pick a point whose fiber is denoted by the central fiber; we are going to study the variation of the complex structure near the central fiber.

The Kodaira–Spencer map has two constructions: the first one is algebraic and can be done over any field, and the second is a geometric construction over the complex numbers. We have the fundamental exact sequence

$$0 \to T_{X_0} \to T_X|_{X_0} \to f^*T_Y|_{\{0\}} \to 0.$$

For simplicity, we assume that the base is a curve. Then we see that $f^*T_Y|_{\{0\}}$ is a one-dimensional vector space spanned by $\mathbb{C}\frac{\partial}{\partial t}$. Giving a lifting map $s :\ f^*T_Y|_{\{0\}} \to T_X\Big|_{X_0}$ is the same as giving a holomorphic vector field $v \in H^0\left(X_0, T_X|_{X_0}\right)$ such that

$$s\left(\frac{\partial}{\partial t}\right) = v.$$

For this to be a splitting, we need

$$df(v) = \frac{\partial}{\partial t}.$$

Then every vector field $\xi \in T_X|_{X_0}$ decomposes uniquely as

$$\xi = (\xi - df(\xi)v) + df(\xi)v.$$

Here

$$df(\xi - df(\xi)v) = df(\xi) - df(\xi)df(v) = 0,$$

so

$$\xi - df(\xi)v \in T_{X_0}.$$

Thus $s :\ f^*T_Y|_{\{0\}} \to T_X\Big|_{X_0}$ induces a holomorphic splitting of the exact sequence.

The obstruction to lifting this holomorphic vector field is determined by the extension class/group. By taking the long exact sequence

$$0 \to H^0(X_0, T_{X_0}) \to H^0(X_0, T_X|_{X_0}) \to H^0(X_0, f^*T_Y|_{\{0\}}) \xrightarrow{\delta} H^1(X_0, T_{X_0}) \to \cdots$$

induced by $(*)$, we know that

$$\tau = \delta(\partial_t) = 0 \iff \text{the exact sequence } (*) \text{ splits.}$$

We call τ the obstruction class, and $H^1(X_0, T_{X_0}) = H^0(X_0, \mathrm{Hom}(f^*T_Y|_{\{0\}}, T_X))$ is called the extension group.

When we talk about the Higgs field or local Torelli, we will use this characterization, which is characteristic-free.

Next, we give a geometric characterization of the Kodaira–Spencer map. Geometrically, we can also construct such a class. Let

$$f : X_\Delta \to \Delta.$$

Then there is an open cover

$$X_\Delta = \bigcup \mathcal{U}_i$$

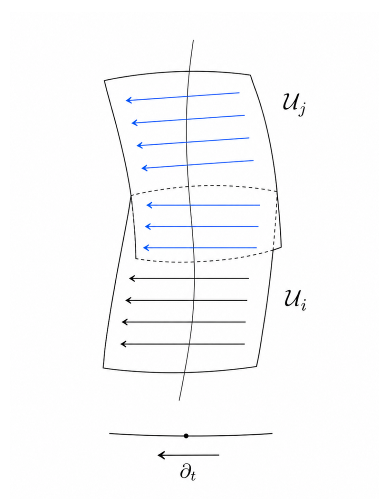
such that over any open neighborhood we have

$$\begin{array}{ccc} \mathcal{U}_i & \xrightarrow{\simeq} & U_i \times \Delta \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

where $U_i = \mathcal{U}_i \cap X_0$, and $\simeq$ above is biholomorphic. Locally on $\mathcal{U}_i$, we have local holomorphic liftings, say

$$u_i \in H^0(\mathcal{U}_i, T_{X_\Delta}), \qquad u_j \in H^0(\mathcal{U}_j, T_{X_\Delta}).$$

The picture is as follows:



Restricting to the central fiber gives

$$u_i|_{U_i} \in H^0\left(U_i, T_{X_\Delta}|_{X_0}\right).$$

By construction,

$$df\left(u_j - u_i\right) = df\left(u_j\right) - df\left(u_i\right) = 0.$$

Hence the difference is a vertical vector field. Therefore

$$\left(u_j - u_i\right)|_{U_i \cap U_j} \in H^0\left(U_i \cap U_j, T_{X_0}\right)$$

lies inside T_{X_0}, not merely in the sheaf of sections of $T_{X_\Delta}|_{X_0}$. Since $\left(u_j - u_i\right)|_{U_i \cap U_j} \in H^0\left(U_i \cap U_j, T_{X_0}\right)$ satisfies the cocycle condition,

$$\rho\left(\frac{\partial}{\partial t}\right) = [\{u_j - u_i\}] \in H^1\left(X_0, T_{X_0}\right)$$

gives the Kodaira–Spencer deformation class along $\frac{\partial}{\partial t}$.

Proposition 1. The holomorphic vector field $\frac{\partial}{\partial t}$ lifts to a horizontal holomorphic vector field if and only if $\rho(\frac{\partial}{\partial t}) = 0$.

Proof. Consider the following fundamental exact sequence

$$0 \to T_{X_0} \to T_X|_{X_0} \xrightarrow{df} f^*T_Y\Big|_{\{0\}} \to 0.$$

Then the Kodaira–Spencer class is just the extension class of this short exact sequence. Since the exact sequence splits if and only if the extension class is 0, this is equivalent to $\rho(\frac{\partial}{\partial t}) = 0$. ∎

We now consider the global Kodaira–Spencer map

$$\tau : T_Y \to R^1 f_* T_{X/Y},$$

which is the first connecting homomorphism of f_* applied to the fundamental exact sequence

$$0 \to T_{X/Y} \to T_X \to f^*T_Y \to 0.$$

We can rewrite the derivative θ of the period mapping of the Legendre family in terms of the Kodaira–Spencer map and the cup product.

$$\begin{array}{ccc} T_{\mathbb{P}^1}(\log S) \otimes R^0 f_* \Omega^1_{E/\mathbb{P}^1}(\log \Delta) & \xrightarrow{\quad\theta\quad} & R^1 f_* \mathcal{O}_E \\ \downarrow{\scriptstyle \tau \otimes id} & \nearrow{\scriptstyle \cup} & \\ R^1 f_* T_{E/\mathbb{P}^1}(-\log S) \otimes R^0 f_* \Omega^1_{E/\mathbb{P}^1}(\log \Delta) & & \end{array}$$

For the Legendre family, τ is an isomorphism, because $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the fine moduli space of elliptic curves with level-two structure. Therefore the tangent space at a point $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is naturally identified with

$$T_{Y,\lambda} \simeq H^1\left(E_\lambda, T_{E_\lambda}\right),$$

and hence the Kodaira–Spencer map is an isomorphism. Note that $\cup$ is an isomorphism because it is simply multiplication by $\mathbb{C}$.