A brief introduction to Algebraic spaces Summer 2025

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The aim of this noe is to give a brief introduction to Algebraic spaces. This note is based on the book [Alp25].

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1 What is Algebraic Space?

2 Criteria for Algebraic Space

2.1 Representable of Diagonal

Theorem 1. (1) The diagonal of an algebraic space is representable by schemes.
(2) The diagonal of an algebraic stack is representable.

Theorem 2. (1) If the diagonal of a stack $\mathcal{X}$ is representable (resp., representable by a scheme), then every morphism $U \to \mathcal{X}$ from a scheme is representable (resp., representable by a scheme).
(2) Every morphism from a scheme to an algebraic stack (resp., algebraic spaces) is representable (resp., representable by schemes).

2.2 Algebraicity of Quotients by Groupoids

Theorem 3. (1) If $R \rightrightarrows U$ is an etale (resp., smooth) groupoid of algebraic spaces. Then $[U/R]$ is a Deligne-Mumford stack (resp., algebraic stack) and $U \to [U/R]$ is an etale (resp., smooth) presentation.
(2) If $R \rightrightarrows U$ be an etale equivalence relation of schemes, then U/R is an algebraic space and $U \to U/R$ is an etale presentation.

Theorem 4. (1) If X is a sheaf on $\mathrm{Sch}_{\mathrm{et}}$ such that there exists a surjective, étale (resp., smooth), and representable morphism $U \to X$ from an algebraic space, then X is an algebraic space.
(2) If $R \rightrightarrows U$ is an etale (resp. smooth) equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.

2.3 Characterization of Algebraic Spaces

Theorem 5. Let $\mathcal{X}$ be an algebraic stack whose diagonal is representable by schemes. The following are equivalent:
(1) the stack $\mathcal{X}$ is an algebraic space,
(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and
(3) every point of $\mathcal{X}$ has a trivial stabilizer.

Theorem 6. For an algebraic stack $\mathcal{X}$, the following are equivalent:
(1) the stack $\mathcal{X}$ is an algebraic space,
(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and
(3) every point of $\mathcal{X}$ has a trivial stabilizer.

3 Examples

References

[Alp25] J. Alper, *tacks and moduli*, 2025, https://sites.math.washington.edu/~jarod/moduli.pdf.