

A brief introduction to Deligne-Mumford Stacks

Summer 2025

Note 3 — 04, 06, 2024 (draft version)

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The aim of this note is to give a brief introduction to Deligne-Mumford stacks. This note is based on the book [Alp25].

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1 What is Deligne-Mumford Stack?

2 Characterization of DM stacks

Theorem 1. Let \mathcal{X} be an algebraic stack. The following are equivalent:

- (1) the stack \mathcal{X} is a Deligne-Mumford;
- (2) the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is unramified; and
- (3) every point of \mathcal{X} has a discrete and reduced stabilizer group.

If \mathcal{X} has quasi-compact diagonal (e.g., \mathcal{X} is quasi-separated), then (3) is equivalent to requiring that every stabilizer is finite and reduced.

Theorem 2. Let $G \rightarrow S$ be a smooth affine group scheme acting on an algebraic space U over S . Then (1) $[U/G]$ is Deligne-Mumford if and only if every point of U has a discrete and reduced stabilizer group, or equivalently if and only if the action map $G \times U \rightarrow U \times U$ is unramified.

(2) $[U/G]$ is an algebraic space if and only if every point of U has a trivial stabilizer group, or equivalently if and only if the action map $G \times U \rightarrow U \times U$ is a monomorphism.

3 Local Structure of Deligne-Mumford Stacks

Theorem 3. Let \mathcal{X} be a quasi-separated Deligne-Mumford stack and $x \in |\mathcal{X}|$ be a finite type point with geometric stabilizer G_x . There exists an étale morphism

$$f : ([\mathrm{Spec} A/G_x], w) \rightarrow (\mathcal{X}, x)$$

where $w \in [\mathrm{Spec} A/G_x]$ such that f induces an isomorphism of geometric stabilizer groups at w . Moreover, if \mathcal{X} has separated (resp., affine) diagonal, then it can be arranged that f is representable and separated (resp., affine).

4 Coarse Moduli

5 Keel-Mori Theorem

5.1 Quotient stack of affine scheme by finite group defines a coarser moduli

5.2 Luna's étale slice theorem

5.3 Descending étale morphism to geometric quotient

5.4 Keel-Mori Theorem

Theorem 4. Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S . Then there exists a coarse moduli space $\pi : \mathcal{X} \rightarrow X$ with $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ such that

- (1) X is separated and of finite type over S ,
- (2) π is a proper universal homeomorphism, and
- (3) for every flat morphism $X' \rightarrow X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \rightarrow X'$ is a coarse moduli space. Moreover, X is proper if and only if \mathcal{X} is.

6 Geometry of Deligne-Mumford Stack

6.1 Criterion for vector bundle on DM stack descent to coarse moduli

6.2 Deligne-Mumford stack has a finite cover by scheme

Theorem 5. (1) Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S . Then there exists a finite, generically étale, and surjective morphism $Z \rightarrow \mathcal{X}$ from a scheme Z .

(2) Let X be a normal algebraic space of finite type over a noetherian scheme S . Show that there is a normal scheme U with an action of a finite abstract group G such that X is the quotient of U by G , i.e., $[U/G] \rightarrow X$ is a coarse moduli space.

6.3 Zariski Main theorem

6.4 Chow Lemma

7 Examples

7.1 Moduli of smooth curves as a Deligne-Mumford Stack

References

- [Alp25] J. Alper, *tacks and moduli*, 2025, <https://sites.math.washington.edu/~jarod/moduli.pdf>.